Some of my favourite math problems

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Here is a collection of some of my favourite math problems and puzzles, in no particular order. I have used all of these problems in classes in the past; they range from "standard" to "nearly impossible," but students always have a lot of fun regardless.

(1) 2022 lockers are numbered 1 through 2022 in that order. Someone goes through and opens every locker. A second person then goes through and closes every second locker. A third person then goes through and changes every 3rd locker: if the locker was open, they close it, and if it was closed, they open it. The 4th person changes every 4th locker. This process continues in the same pattern for 2022 people. After the 2022nd person has completed their task, which lockers are left open?

(2) If you draw n points on the circumference of a circle and join all possible chords between them, what is the maximum number of regions into which the chords partition the circle?

(3) In each of seven boxes $B_1, B_2, B_3, B_4, B_5, B_6, B_7$, there is initially one stone. You are allowed two types of moves:

(T1) Remove one stone from a nonempty box B_i with $i \leq 6$ and add two stones to box B_{i+1}.

(T2) Remove one stone from a nonempty box B_j with $j \leq 5$ and swap the (possibly equal) contents of (possibly empty) boxes B_{j+1} and B_{j+2}.

What is the maximum number of stones possible in box B_7 after any sequence of moves?

(4) A monk begins at the base of a mountain at 8AM and takes the single, narrow, winding path up to the temple at the peak and arrives at 8PM. The next day, the monk begins to travel down the mountain starting at 8AM using the same route, and arrives at the base of the mountain at 8PM. Prove that there is some point along the path that the monk visited at the same time on both days.

(5) Choose an integer $n \geq 2$. Starting with 1, on the first step add $1/n$, and on subsequent steps either add $1/n$ or take the reciprocal. What is the fewest number of steps required to return to 1? Try this for $n = 6$, $n = 11$, then general n.

(6) How do you hang a picture on a wall with a string using two nails in such a way that removing either of the nails will make the picture and string fall down?

(7) How many triangles of nonzero area can be formed whose vertices come from an $m \times n$ grid? Try this for $m = n = 3, 4, 5$, then general m, n.

(8) Prisoners labelled $1, 2, \ldots$ are lined up and facing in one direction. Each is given a hat that is either red or blue. Due to their lineup, Prisoner 1 can see the rest of the prisoners' hats, Prisoner 2 can see the hats of prisoners 3 and on, Prisoner 3 can see the hats of prisoners 4 and on, and so forth. They must all guess the colour of their own hat at the same time without hearing what other prisoners have to say. If the prisoners are allowed to meet beforehand to discuss strategies, does there exist a way to guarantee all but finitely many prisoners guess correctly? Assume the prisoners have infinite memory and can process the infinitely many colours of hats in front of them instantly.

(9) Find positive integer values for 🍎, 🍌, and 🍍 such that

$$\frac{🍎}{🍌 + 🍍} + \frac{🍌}{🍍 + 🍎} + \frac{🍍}{🍎 + 🍌} = 4$$

(10) Does there exist a way to fill a 3×3 table with nine distinct perfect square numbers such that every row, column, and two diagonals sum to the same number?

(11) What is the minimal area of a region in the plane in which a needle of unit length can be turned around, allowing translations and rotations of the needle? Assume the needle has no width.

(12) Suppose $a < b$ are two different whole numbers greater than 1 with sum no greater than 100. Susan knows the sum $a + b$ and Peter knows the product ab. Both Susan and Peter know all of this information. The following conversation occurs, with both participants telling the truth:

- Peter says "I don't know the numbers."
- Susan says "I knew that already."
- Peter says "Now I know a and b."
- Susan says "Now I also know a and b."

What are a and b?

(13) A mathematician has 5 atoms of a radioactive isotope. Every day, each atom has a probability $1/2$ of decaying. What is the expected number of days until all atoms have decayed?

(14) Is $\pi^{\pi^{\pi^{\pi}}}$ an integer?

(15) Suppose you have n pairs of boxes distributed in a rectangle such that each pair has the same label. Although no two boxes overlap with each other or cross the boundary of the rectangle, some of these boxes may be flush with an edge of the rectangle and some may not. Under what condition will there exist nonintersecting lines which join each pair such that no line passes through a box or outside the rectangle?