Euler's Constant

In this problem, we will investigate the partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k}$ as $n \to \infty$. In particular, we aim to show that the sums behave like the natural logarithm, by showing that there exists a constant γ such that

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = \gamma$$

The constant γ is called *Euler's constant.*

(a) Let

$$T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n.$$

Evaluate T_n for $n = 1, 2, 3, 4, 5$.

(b) For T_n defined as in part (a), show the sequence $\{T_n\}$ converges by following the following steps:

(i) Show the sequence $\{T_n\}$ is monotone decreasing. (Hint: Show that $\ln(1 + \frac{1}{n}) > \frac{1}{n+1}$.)

(ii) Show the sequence $\{T_n\}$ is bounded below. (Hint: Express $\ln n$ as a definite integral.)

(iii) Conclude that $\{T_n\}$ converges.

(c) Now estimate how far T_n is from γ for a given integer n.

(i) Show that $\ln(n + 1) - \ln n < \frac{1}{n}$

(ii) Show that for any integer n, $T_n - T_{n+1} < \frac{1}{n} - \frac{1}{n+1}$

(iii) Show that for any integers n and m with $n < m$, $0 < T_n - T_m < \frac{1}{n} - \frac{1}{m}$. (Hint: Make a telescoping sum.)

(iv) Show that $0 \le T_n - \gamma \le \frac{1}{n}$.

(v) Use a computer algebra system to estimate γ to an accuracy of within 0.001.