Basel Problem

In this problem, we will calculate $\sum_{n=1}^{\infty} \frac{1}{k^2}$.

(a) For $0 < x < \pi/2$, prove

$$\frac{1}{\sin^2 x} - 1 < \frac{1}{x^2} < \frac{1}{\sin^2 x}$$

(b) Partition the interval $[0, \pi/2]$ into 2^n equal parts, and let $x_k = k \cdot \frac{\pi/2}{2^n}$. Define

$$S_n = \sum_{k=1}^{2^n-1} \frac{1}{\sin x_k^2}.$$

Prove that

$$S_n - (2^n - 1) < \frac{4^{n+1}}{\pi^2} \sum_{k=1}^{2^n-1} \frac{1}{k^2} < S_n.$$

(c) Prove

$$\frac{1}{\sin^2 x} + \frac{1}{\sin^2(\frac{\pi}{2} - x)} = \frac{4}{\sin^2 2x}.$$

(d) Prove

$$S_n = 4S_{n-1} + 2.$$

(e) Prove

$$S_n = \frac{2(4^n - 1)}{3}.$$

(f) Find $\sum_{n=1}^{\infty} \frac{1}{k^2}$.