An improper integral

Does $\displaystyle\int_0^\infty \sin x \sin x^2\,dx$ converge? Why or why not?

Solution: Method 1: Integrating by parts with $f = \frac{\sin x}{2x}$ and $g' = 2x\sin x^2$, we get

$$\begin{aligned}
\int_0^b \sin x \sin x^2\,dx &= \int_0^b \frac{\sin x}{2x}\cdot 2x\sin x^2\,dx \\
&= -\cos x^2 \cdot \frac{\sin x}{2x}\Big|_0^b + \int_0^b \cos x^2 \cdot \frac{\cos x(2x) - 2\sin x}{(2x)^2}\,dx \\
&= -\cos x^2 \cdot \frac{\sin x}{2x}\Big|_0^b + \int_0^b \cos x^2 \cdot \frac{\cos x}{2x}\,dx - \int_0^b \cos x^2 \cdot \frac{\sin x}{2x^2}\,dx
\end{aligned}$$

Method 2: