Unit 1

A graphing calculator is required for these questions.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

1. The number of books sold in print of a new nonfiction book can be modeled by a function $N(t)$, given below, where t is the number of months since publication and h is a continuous function for $0 \le t \le 10$. Selected values of h are given in the table.

$$N(t) = \begin{cases} h(t), & 0 \le t \le 10 \\ \dfrac{8700t^2}{3t^2+100}, & t > 10 \end{cases}$$

t (months)	0	1	3	4	8	10
$h(t)$ (books)	0	455	780	1340	1915	2175

(a) Find the average rate of change of N between $t = 1$ and $t = 4$. Indicate units of measure.

Solution: The average rate of change is

$$\frac{N(4) - N(1)}{4 - 1} = \frac{1340 - 455}{3} = \frac{885}{3} = 295$$

So the average rate of change is 295 books per month.

(b) Must there be a time t, for $0 \le t \le 10$, when $N(t) = 1000$? Justify your answer.

Solution: Yes. Since $N(t) = h(t)$ on $[0, 10]$ and h is continuous, we can apply the Intermediate Value Theorem. We have $N(4) = 1340 > 1000$ and $N(3) = 780 < 1000$. Since 1000 is between 780 and 1340, by the IVT there must exist some t in $(3, 4)$ such that $N(t) = 1000$.

(c) Find $\lim_{t\to\infty} N(t)$. Explain the meaning of $\lim_{t\to\infty} N(t)$ in the context of this problem.

Solution:

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{8700t^2}{3t^2+100} = \lim_{t\to\infty} \frac{8700}{3 + \frac{100}{t^2}} = \frac{8700}{3} = 2900$$

This means that as time goes on (as the number of months since publication becomes very large), the number of books sold approaches 2900 books. In other words, the book will eventually sell approximately 2900 copies in print.

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

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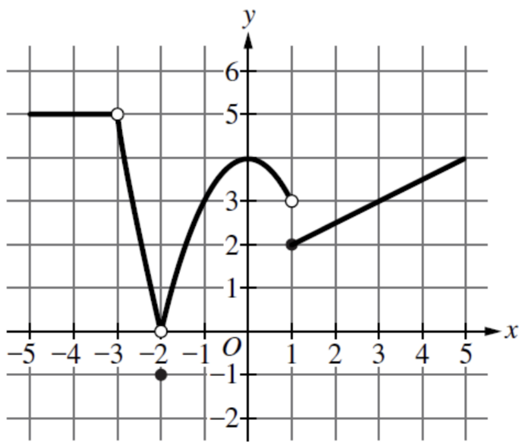
2. The graph of a function f is shown in the xy-plane for $-5 \le x \le 5$.



Let

$$g(x) = \begin{cases} 3\sin\left(\frac{\pi x}{4}\right) - 1, & x \le 2 \\ kx - 7, & x > 2 \end{cases}$$

for some constant k.

(a) For each of the values $a = -3$, $a = 0$, and $a = 1$, find $\lim_{x\to a} f(x)$, or explain why it does not exist.

Solution: $\lim_{x\to -3} f(x) = 5$, $\lim_{x\to 0} f(x) = 4$, $\lim_{x\to 1} f(x) = DNE$ (the left and right limits are not the same).

(b) Let $h(x) = 8f(x) - 5g(x)$. Find $\lim_{x\to -2} h(x)$.

Solution: From the graph, $\lim_{x\to -2} f(x) = 0$.

For $x < 2$, g is continuous, so: $\lim_{x\to -2} g(x) = 3\sin\left(\frac{-2\pi}{4}\right) - 1 = 3\sin\left(-\frac{\pi}{2}\right) - 1 = 3(-1) - 1 = -4.$

Then $\lim_{x\to -2} h(x) = 8 \lim_{x\to -2} f(x) - 5 \lim_{x\to -2} g(x) = 8(0) - 5(-4) = 20$

(c) Find the value of k so that g is continuous at $x = 2$. Justify your answer.

Solution: For g to be continuous at $x = 2$, we need $\lim_{x\to 2} g(x) = g(2)$.

First, find $g(2)$ using the top piece (since $x \le 2$):

$$g(2) = 3\sin\left(\frac{2\pi}{4}\right) - 1 = 3\sin\left(\frac{\pi}{2}\right) - 1 = 3(1) - 1 = 2$$

This is also the value of $\lim_{x\to 2^-} g(x)$ by continuity.

Find the right-hand limit (using the bottom piece):

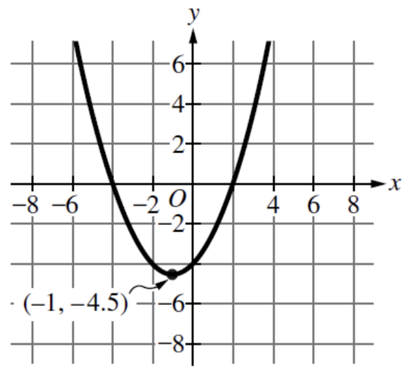
$$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (kx - 7) = 2k - 7$$

For continuity, we need:

$$2k - 7 = 2$$
$$k = \frac{9}{2}$$

USE OF A CALCULATOR IS NOT ALLOWED FOR THIS QUESTION.

3. The graph of a quadratic function f is shown in the xy-coordinate plane.



(a) Which is greater: the average rate of change of f over the interval $[-3, 2]$ or the instantaneous rate of change of f at $x = 2$? Give a reason for your answer.

Solution: The instantaneous rate of change of f at $x = 2$ is greater as the slope is steeper.

(b) Find $\lim_{x \to 0^-} [-2f(x) + 9]$.

Solution: $\lim_{x \to 0^-} [-2f(x) + 9] = -2(-4) + 9 = 17$

(c) Let $j(x) = \dfrac{x^2 - 7x + 10}{f(x)}$. Find all values of x where j has a discontinuity due to a vertical asymptote.

Solution: We look for where j goes to ∞ or $-\infty$ either approaching from the left or right. Candidates are where $f(x) = 0$, that is, $x = -4, 2$. However, since the numerator and denominator are both quadratics without repeated roots, we only get a vertical asymptote when the numerator is also not 0. Since $x = 2$ makes the numerator zero $(2^2 - 7(2) + 10 = 4 - 14 + 10 = 0)$, $x = 2$ gives a removable discontinuity, not a vertical asymptote. Thus, only $x = -4$ gives a vertical asymptote.

(d) Let $g(x) = \dfrac{f(x)}{x+4}$. Find $\lim_{x \to -4} g(x)$.

Solution: We can see that $f(x) = a(x+4)(x-2)$ by the zeroes of the graph. Since $(-1, -4.5)$ is on the graph, we get

$$-4.5 = a(-1+4)(-1-2) = a(3)(-3) = -9a \implies a = \frac{1}{2}.$$

Then

$$\lim_{x \to -4} g(x) = \lim_{x \to -4} \frac{\frac{1}{2}(x+4)(x-2)}{x+4} = \lim_{x \to -4} \frac{1}{2}(x-2) = \frac{1}{2}(-4-2) = -3$$

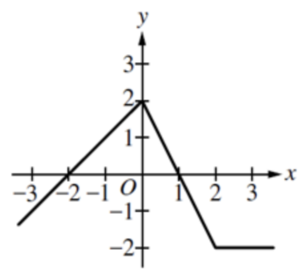
Unit 2

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

1.

x	$f(x)$	$f'(x)$
-2	2	-1
-1	0	0
0	1	1
1	2	$\frac{1}{2}$
2	4	2



Graph of $g(x)$

$f(x)$ and $g(x)$ are two continuous differentiable functions. Selected values of $f(x)$ and $f'(x)$ are given above. The graph of $g(x)$ is piecewise linear and made up of three line segments, as shown above.

(a) Find the average rate of change for $f(x)$ on the closed interval $[-2, 2]$.

Solution: $\dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{4 - 2}{4} = \dfrac{1}{2}$

(b) Estimate the instantaneous rate of change of $f(x)$ at $x = 1/2$.

Solution: $f'(1/2) \approx \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{2 - 1}{1} = 1$

(c) Find the average rate of change for $g(x)$ on the closed interval $[-2, 2]$.

Solution: $\dfrac{g(2) - g(-2)}{2 - (-2)} = \dfrac{-2 - 0}{4} = -\dfrac{1}{2}$

(d) Find the instantaneous rate of change of $g(x)$ at $x = 1/2$.

Solution: Using the slope of the line, $g'(1/2) = \dfrac{-2 - 2}{2 - 0} = \dfrac{-4}{2} = -2$

(e) Find $\dfrac{d}{dx}[f(x) \cdot g(x)]_{x=1}$

Solution: By the product rule, $\dfrac{d}{dx}[f(x) \cdot g(x)]_{x=1} = f'(1)g(1) + f(1)g'(1) = \dfrac{1}{2} \cdot 0 + 2 \cdot (-2) = -4$

(f) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

Solution: By the Quotient Rule, $h'(1) = \dfrac{g'(1)f(1) - g(1)f'(1)}{[f(1)]^2} = \dfrac{(-2) \cdot 2 - 0 \cdot \frac{1}{2}}{2^2} = -1$

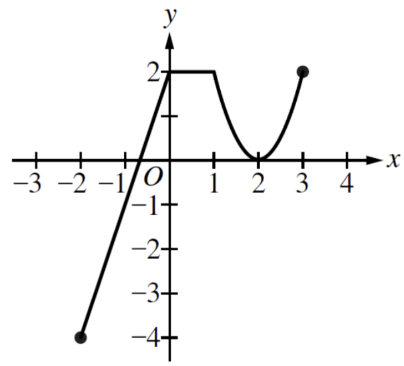
(g) Find $g'(-1)$ and $g'(5/2)$ or state if the values do not exist.

Solution: Using the slope of the line, $g'(-1) = \dfrac{2 - 0}{0 - (-2)} = \dfrac{2}{2} = 1$ and $g'(5/2) = 0$.

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

2.



Let $R(x) = x^2 + 3e^x$ and $S(x)$ be the continuous function shown, consisting of two line segments and the curve $y = 2(x-2)^2$ for $1 \le x \le 3$. (You may use a calculator, but for best practice, try to formulate your responses without one first—and then check yourself with your device!)

(a) Use the definition of the derivative to set up (but not evaluate) a limit that would find $R'(x)$ in terms of x.

Solution:

$$R'(x) = \lim_{h\to 0} \frac{R(x+h) - R(x)}{h} = \lim_{h\to 0} \frac{((x+h)^2 + 3e^{x+h}) - (x^2 + 3e^x)}{h}$$

(b) Find $\frac{d}{dx}[R(x)\cdot S(x)]_{x=2.5}$ to three decimal places.

Solution: By the product rule and using $R'(x) = 2x + 3e^x$,

$$\frac{d}{dx}[R(x)\cdot S(x)]_{x=2.5} = R'(2.5)S(2.5) + R(2.5)S'(2.5) \approx 41.54748 \cdot 0.5 + 42.79748 \cdot 2 \approx 106.369$$

(c) Let $T(x) = \sin x \cdot R(x)$. Find $T'(5)$ to three decimal places.

Solution: Using the calculator's numeric derivative feature, we immediately get:

$$T'(5) \approx -303.151$$

(d) Let $U(x) = \dfrac{\tan x}{S(x)}$. Find $U'(x)$ and find $\left.\dfrac{dU}{dx}\right|_{x=\pi/4}$.

Solution: By the Quotient Rule,

$$U'(x) = \frac{\sec^2 x \cdot S(x) - \tan x \cdot S'(x)}{[S(x)]^2}$$

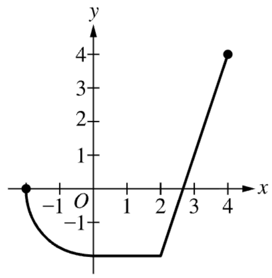
so

$$\left.\frac{dU}{dx}\right|_{x=\pi/4} = \frac{\sec^2 \frac{\pi}{4}\cdot S(\frac{\pi}{4}) - \tan\frac{\pi}{4}\cdot S'(\frac{\pi}{4})}{[S(\frac{\pi}{4})]^2} = \frac{2\cdot 2 - 1\cdot 0}{2^2} = 1$$

USE OF A CALCULATOR IS NOT ALLOWED FOR THIS QUESTION.

3.

x	$n(x)$	$n'(x)$
-2	-2	2
-1	1	1
$\frac{1}{2}$	2	1
1	$\frac{1}{2}$	-2
3	-1	2



Graph of p

$$r(x) = \begin{cases} 2x+4, & x < -1 \\ -x^3+1, & -1 \le x \le 1 \\ x^2+\sin x, & x > 1 \end{cases}$$

Selected values of the twice differentiable function n are given in the table. The function p is the piecewise function made up of the quarter circle $y = \sqrt{4-x^2}$ and two line segments, all as shown in the graph. A third function, r, is defined analytically as a piecewise function.

(a) Use continuity to show that $r(x)$ is not differentiable at $x = 1$.

Solution:

$$\lim_{x\to 1^-} r(x) = \lim_{x\to 1^-}(-x^3+1) = 0 \neq 1+\sin 1 = \lim_{x\to 1^+}(x^2+\sin x) = \lim_{x\to 1^-} r(x)$$

so $r(x)$ is not continuous and hence not differentiable at $x = 1$.

(b) Use the definition of derivative with one-sided limits to show that $p'(2)$ does not exist.

Solution: $\lim_{h\to 0^-} \dfrac{p(2+h)-p(2)}{h} = 0 \neq 3 = \dfrac{4-(-2)}{4-2} = \lim_{h\to 0^+} \dfrac{p(2+h)-p(2)}{h}$ so $p'(2)$ does not exist.

(c) Estimate $n'(2)$.

Solution: $n'(2) \approx \dfrac{n(3)-n(1)}{3-1} = \dfrac{-1-1/2}{2} = -\dfrac{3}{4}$

(d) Find $\dfrac{d}{dx}[p(x)\cdot r(x)]$ for $0 < x < 1$ in terms of x.

Solution: By the product rule,

$$\frac{d}{dx}[p(x)\cdot r(x)] = p'(x)r(x) + p(x)r'(x) = 0\cdot(-x^3+1) + (-2)\cdot(-3x^2) = 6x^2$$

(e) Let $Q(x) = \dfrac{p(x)\cdot r(x)}{n(x)}$. Find $Q'(\frac{1}{2})$.

Solution: By the Product and Quotient Rules,

$$\begin{aligned} Q'(\frac{1}{2}) &= \frac{\frac{d}{dx}[p(x)\cdot r(x)]_{x=1/2}\cdot n(\frac{1}{2}) - p(\frac{1}{2})r(\frac{1}{2})\cdot n'(\frac{1}{2})}{[n(\frac{1}{2})]^2} \\ &= \frac{[p'(\frac{1}{2})\cdot r(\frac{1}{2}) + p(\frac{1}{2})\cdot r'(\frac{1}{2})]\cdot 2 - (-2)(-(\frac{1}{2})^3+1)(1)}{2^2} \\ &= \frac{[0\cdot r(\frac{1}{2}) + (-2)\cdot(-3(\frac{1}{2})^2)]\cdot 2 - (-2)((-\frac{7}{8}))}{4} = \frac{-3-\frac{7}{4}}{4} = -\frac{19}{16} \end{aligned}$$

Unit 3

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

1.

x	$J(x)$	$J'(x)$
2	-5	8
3	-2	1
4	-1	3
5	4	7
6	8	2
7	9	6

(a) Let A be the function defined by $A(x) = J(3x+1)$. Find $A'(2)$.

Solution: By Chain Rule, $A'(2) = J'(3x+1)|_{x=2} \cdot \dfrac{d}{dx}(3x+1) = J'(7) \cdot 3 = 6 \cdot 3 = 18$

(b) Let W be the inverse of J. Find $W'(4)$.

Solution: $W'(x) = \dfrac{1}{J'(W(x))} = \dfrac{1}{J'(5)} = \dfrac{1}{7}$

(c) Consider the curve given by the implicit relation $\sin y = \frac{J(x)}{10}$. It can be shown that the point $(2, \frac{7\pi}{6})$ lies on this curve. Find $\frac{dy}{dx}$ at the point $(2, \frac{7\pi}{6})$.

Solution:

$$\cos y \cdot \frac{dy}{dx} = \frac{J'(x)}{10}$$
$$\left.\frac{dy}{dx}\right|_{(2,\frac{7\pi}{6})} = \frac{J'(2)}{10} \cdot \sec\frac{7\pi}{6} = \frac{8}{10} \cdot \left(-\frac{2}{\sqrt{3}}\right) = -\frac{8}{5\sqrt{3}}$$

(d) The implicit relation in part (c) may be rewritten to give the explicit function $y = \arcsin\left(\frac{J(x)}{10}\right)$. If $u(x) = \frac{J(x)}{10}$, then $y = \arcsin u$. When $x = 2$ and $u(2) = -\frac{1}{2}$, find $\frac{dy}{du}$.

Solution: $\left.\dfrac{dy}{du}\right|_{u=-\frac{1}{2}} = \left.\dfrac{1}{\sqrt{1-u^2}}\right|_{u=-\frac{1}{2}} = \dfrac{1}{\sqrt{1-\frac{1}{4}}} = \dfrac{1}{\sqrt{\frac{3}{4}}} = \dfrac{2}{\sqrt{3}}$

(e) Using the results from part (d), find $\frac{dy}{dx}$ when $x = 2$.

Solution: By the Chain Rule,

$$\left.\frac{dy}{dx}\right|_{x=2} = \left.\frac{dy}{du}\right|_{u=-\frac{1}{2}} \cdot \left.\frac{du}{dx}\right|_{x=2} = \frac{2}{\sqrt{3}} \cdot \frac{J'(2)}{10} = \frac{2}{\sqrt{3}} \cdot \frac{8}{10} = \frac{8}{5\sqrt{3}}$$

(f) Why are the answers to (c) and (e) different?

Solution: (e) requires the restricted domain of $\sin x$ in order for arcsin to be defined. In particular, we need the argument of $\sin x$ to be in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which does not include $\frac{7\pi}{6}$. Thus (c) uses a different branch of the inverse sine function than (e), giving opposite signs for the derivative.

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

2. Consider the function $f(x) = 2(x^3 + 7)^{1/3} + x$.

(a) Find the slope of the line tangent to the graph of f at the point where $x = 1$.

Solution: Using the calculator's numeric derivative feature, we get $f'(1) = 1.5$

(b) Write an expression for $f'(x)$. Use this expression to confirm your answer to part (a). Use a graphing calculator and your $f'(x)$ expression to evaluate the second derivative of f when $x = 1$.

Solution:

$$f'(x) = 2 \cdot \frac{1}{3}(x^3 + 7)^{-2/3} \cdot 3x^2 + 1 = 2x^2(x^3 + 7)^{-2/3} + 1$$

At $x = 1$:

$$f'(1) = 2(1)^2(1 + 7)^{-2/3} + 1 = 2(8)^{-2/3} + 1 = 2\left(\frac{1}{4}\right) + 1 = \frac{1}{2} + 1 = 1.5$$

Using a calculator, $f''(1) = 0.875$

(c) Let $g(x)$ represent the inverse of $f(x)$. Write an equation for the line tangent to the graph of g at $x = 5$.

Solution: Since $f(1) = 2(1 + 7)^{1/3} + 1 = 2(2) + 1 = 5$, $g'(5) = \frac{1}{f'(1)} = \frac{1}{1.5} = \frac{2}{3}$.
The tangent line is:

$$y - 1 = \frac{2}{3}(x - 5)$$

(d) Consider the curve defined by the implicit relation $2y^2 - ye^x = 5x$. Find the slope of the line tangent to the curve at the point where $y = 4$.

Solution: When $y = 4$, we get $x \approx 1.75824$. Using implicit differentiation:

$$4y \cdot \frac{dy}{dx} - \left(\frac{dy}{dx}e^x + ye^x\right) = 5$$

$$(4y - e^x)\frac{dy}{dx} - ye^x = 5$$

$$\frac{dy}{dx} = \frac{5 + ye^x}{4y - e^x}$$

At the point where $y = 4$ and $x \approx 1.75824$:

$$\frac{dy}{dx} = \frac{5 + ye^x}{4y - e^x} \approx \frac{5 + 4 \cdot 5.80220}{4(4) - 5.80220} \approx 2.766$$

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

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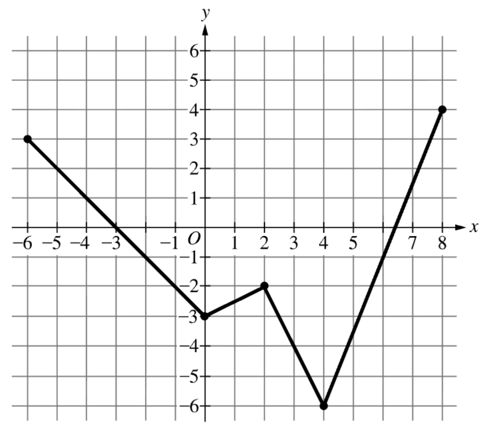
3.



Graph of h

Let g be the function defined by $g(x) = (x^2+2)^{\sin x}$. Let h be the function whose graph, consisting of four line segments, as shown in the figure.

(a) For some function $r(x)$, it is known that $r'(x) = \frac{h(x)}{2x+13}$. Find $r''(-5)$.

Solution: $r''(-5) = \left.\dfrac{h'(x)(2x+13) - 2\cdot h(x)}{(2x+13)^2}\right|_{x=-5} = \dfrac{(-1)(3)-(2)(2)}{3^2} = -\dfrac{7}{9}$

(b) Let q be the function defined by $q(x) = h(g(3x))$. Find $q'(0)$.

Solution:

$$q'(0) = h'(g(3x))\cdot g'(3x)\cdot 3|_{x=0} = h'(g(0))\cdot g'(0)\cdot 3$$
$$= h'(1)\cdot 0.69315\cdot 3 = \frac{1}{2}\cdot 2.07945 = 1.040$$

(c) Let k be the function defined by $k(x) = g(x)\cdot h(2x)$. Find $k'(3)$.

Solution:

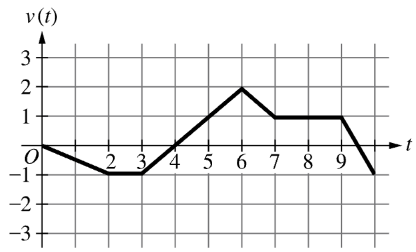
$$k'(3) = g'(x)\cdot h(2x) + g(x)\cdot h'(2x)\cdot 2|_{x=3} = g'(3)\cdot h(6) + g(3)\cdot h'(6)\cdot 2$$
$$= -3.22187\cdot(-1) + 1.40269\cdot\frac{-1-(-6)}{6-4}\cdot 2 = 10.235$$

Unit 4

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

1.



A particle is moving along the x-axis with position $x(t)$ at time t, for $0 \leq t \leq 10$. At time $t = 6$, the particle is at position $x = -3$. The graph of the particle's velocity, v, as a function of time, t, is provided in the figure.

(a) Find the acceleration of the particle at time $t = 1$.

Solution: $a(1) = v'(1) = \dfrac{-1-0}{2-0} = -\dfrac{1}{2}$

(b) Interpret the graph of $v(t)$ to identify all times on the interval $0 \leq t \leq 10$ when the particle changes direction.

Solution: The particle changes direction when $v(t)$ changes sign. From the graph, this occurs at $t = 4$ and $t = 9.5$.

(c) At time $t = 6$, is the particle moving towards the origin or away from the origin?

Solution: At $t = 6$, $v(6) = 2 > 0$ (the velocity is positive), so the particle is moving right. Since $x(6) = -3$ (left of the origin), the particle is moving toward the origin.

(d) Determine $\lim\limits_{t \to 6}(t + 2 \cdot x(t))$ and $\lim\limits_{t \to 6} \sin(\pi t)$. Evaluate $\lim\limits_{t \to 6} \dfrac{t + 2 \cdot x(t)}{\sin(\pi t)}$.

Solution:

$$\lim_{t \to 6}(t + 2x(t)) = 6 + 2(-3) = 0$$

$$\lim_{t \to 6} \sin(\pi t) = \sin(6\pi) = 0$$

Since both limits are 0, we try L'Hôpital's Rule:

$$\lim_{t \to 6} \frac{t + 2x(t)}{\sin(\pi t)} = \lim_{t \to 6} \frac{1 + 2v(t)}{\pi \cos(\pi t)} = \frac{1 + 2v(6)}{\pi \cos(6\pi)} = \frac{1 + 2(2)}{\pi(1)} = \frac{5}{\pi}$$

A GRAPHING CALCULATOR IS REQUIRED FOR THESE QUESTIONS.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

2.

t (hours)	1	2	3	7	8
$M'(t)$ (shirts/hour)	70	82	86	115	121

Designs by AP makes and sells shirts. The number of shirts that have been produced at time t hours after the start of the work-day can be modeled by $M(t)$, where M is twice-differentiable and t is measured in hours for $0 \le t \le 8$. The table gives values of $M'(t)$, the rate of change in M, at selected times during the workday.

(a) Approximate $M''(t)$ using the average rate of change of $M'(t)$ over the interval. Using correct units, interpret the meaning of $M''(5)$ in the context of this problem.

Solution:

$$M''(5) \approx \frac{M'(7) - M'(3)}{7-3} = \frac{115-86}{4} = \frac{29}{4} = 7.25 \text{ shirts/hour}^2$$

$M''(5)$ represents the rate at which the production rate is increasing at $t = 5$ hours.

(b) The total number of shirts that have been produced by time $t = 3$ is 225. Approximate $M(3.5)$ using the equation of the line tangent to the graph of M at $t = 3$. It is known that $M'(t)$ is increasing on $0 \le t \le 8$. Is the approximation an overestimate or underestimate of the total number of shirts produced at time $t = 3.5$?

Solution: The tangent line at $t = 3$: $y = M(3) + M'(3)(x-3) = 225 + 86(x-3)$, so $M(3.5) \approx 225 + 86(0.5) = 225 + 43 = 268$.

Since $M'(t)$ is increasing, the graph of M is concave up, so the tangent line lies below the graph. Therefore, the approximation is an underestimate.

(c) *Designs by AP* sells their shirts online. The number of shirts that have been sold online at time t hours after the start of the workday can be modeled by $S(t)$, where S is twice-differentiable and t is measured in hours for $0 \le t \le 8$. The rate at which shirts are sold, $S'(t)$, is given by the equation $S'(t) = 20\sqrt{t} - t^2 + 55 \cdot \arctan(t^3)$. Determine all time intervals, on $0 \le t \le 8$, for which the rate of sales is at least 100 shirts per hour.

Solution: Using a graphing calculator, solve $S'(t) \ge 100$, we get $1.76723 \le t \le 5.88617$.

(d) Calculate $M'(3) - S'(3)$. Using correct units, interpret the meaning of $M'(3) - S'(3)$ in the context of this problem.

Solution:

$$M'(3) = 86$$

$$S'(3) = 20\sqrt{3} - 9 + 55\arctan(27) \approx 34.641 - 9 + 55(1.535) = 34.641 - 9 + 84.425 = 110.066$$

$$M'(3) - S'(3) \approx 86 - 110.066 = -24.066$$

This means that at $t = 3$ hours, shirts are being sold faster than they are being produced by approximately 24 shirts per hour. The inventory of shirts is decreasing at that time.

NO CALCULATOR IS ALLOWED FOR THESE QUESTIONS.

Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

3. Ayan is driving due east and passes through an intersection at time $t = 0$ seconds. The velocity of Ayan's car can be modeled by the equation $v(t) = -\frac{1}{10}t^3 + \frac{3}{2}t^2 - 4t + 10$, for $0 \le t \le 11$, where $v(t)$ is measured in meters per second and t is measured in seconds. Ayan's car is 900 meters east of the intersection after 10 seconds.

(a) Find the acceleration of Ayan's car at time $t = 10$. Indicate units of measure.

> **Solution:** $a(t) = v'(t) = -\frac{3}{10}t^2 + 3t - 4$
> At $t = 10$: $a(10) = -\frac{3}{10}(10)^2 + 3(10) - 4 = -\frac{3}{10}(100) + 30 - 4 = -30 + 30 - 4 = -4 \text{ m/s}^2$

(b) Is Ayan's car speeding up or slowing down at time $t = 10$?

> **Solution:** $v(10) = -\frac{1}{10}(10)^3 + \frac{3}{2}(10)^2 - 4(10) + 10 = -100 + 150 - 40 + 10 = 20$ m/s. Since $v(10) > 0$ and $a(10) < 0$, the car is slowing down (velocity and acceleration have opposite signs).

(c) Keng is standing 1200 meters due north of the intersection. Find the rate of change of the distance between Keng and Ayan, in meters per second, at time $t = 10$.

> **Solution:** Let $A(t)$ be Ayan's position east of the intersection and $K(t)$ be the distance between Keng and Ayan. We have $A(10) = 900$ and $A'(10) = v(10) = 20$.
> Using the Pythagorean theorem: $(K(t))^2 = (A(t))^2 + (1200)^2$
> Differentiating: $2K(t)K'(t) = 2A(t)A'(t)$
> At $t = 10$, $K'(10) = \dfrac{900(20)}{\sqrt{900^2 + 1200^2}} = \dfrac{18000}{\sqrt{810000 + 1440000}} = \dfrac{18000}{\sqrt{2250000}} = \dfrac{18000}{1500} = 12$
> So the distance between Keng and Ayan is increasing at 12 m/s.

(d) Ben is in a hot air balloon directly over the intersection. Let θ be the angle of elevation from Ayan to Ben, where θ is measured in radians. Let A be the distance between Ayan's car and the intersection and B be the distance between Ben and the intersection at time t, both measured in meters. These quantities satisfy the relationship $\tan\theta = \frac{B}{A}$. Write an equation that describes the relationship between the rates of change of θ, A, and B.

> **Solution:**
>
> $$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{A \cdot \frac{dB}{dt} - B \cdot \frac{dA}{dt}}{A^2}$$

(e) Using θ as defined in Part D, at time $t = 10$, $\frac{d\theta}{dt} = -0.0073$. Using correct units, interpret the meaning of $\frac{d\theta}{dt}$ in the context of this problem.

> **Solution:** $\frac{d\theta}{dt} = -0.0073$ radians per second means that at $t = 10$ seconds, the angle of elevation from Ayan's car to Ben is decreasing at a rate of 0.0073 radians per second. In other words, as Ayan drives further east away from the intersection, the angle from his car up to the balloon is getting smaller.

Answer Key

Unit 1

1a) 295 books per month

1b) Yes, by IVT on $[3,4]$ since $N(3) = 780 < 1000 < N(4) = 1340$.

1c) 2900 books (horizontal asymptote)

2a) $\lim_{x\to -3} f(x) = 5, \quad \lim_{x\to 0} f(x) = 4, \quad \lim_{x\to 1} f(x) = \text{DNE}$

2b) 20

2c) $k = \frac{9}{2}$

3a) Instantaneous at $x = 2$ is greater (steeper slope)

3b) 17

3c) $x = -4$ only

3d) -3

Unit 2

1a) $\frac{1}{2}$

1b) 1

1c) $-\frac{1}{2}$

1d) -2

1e) -4

1f) -1

1g) $g'(-1) = 1, \quad g'(5/2) = 0$

2a) $\lim\limits_{h\to 0} \dfrac{((x+h)^2 + 3e^{x+h}) - (x^2 + 3e^x)}{h}$

2b) 106.369

2c) -303.151

2d) $U'(x) = \dfrac{\sec^2 x \cdot S(x) - \tan x \cdot S'(x)}{[S(x)]^2}; \quad U'(\pi/4) = 1$

3a) Not continuous at $x = 1$: $0 \neq 1 + \sin 1$

3b) Left derivative 0, right derivative 3

3c) $-\frac{3}{4}$

3d) $6x^2$

3e) $-\frac{19}{16}$

Unit 3

1a) 18

1b) $\frac{1}{7}$

1c) $-\dfrac{8}{5\sqrt{3}}$

1d) $\dfrac{2}{\sqrt{3}}$

1e) $\dfrac{8}{5\sqrt{3}}$

1f) Different branches of inverse sine (principal vs. $\frac{7\pi}{6}$)

2a) 1.5

2b) $f'(x) = 2x^2(x^3+7)^{-2/3} + 1; \quad f''(1) = 0.875$

2c) $y - 1 = \dfrac{2}{3}(x-5)$

2d) 2.766

3a) $-\frac{7}{9}$

3b) 1.040

3c) 10.235

Unit 4

1a) $-\dfrac{1}{2}$ m/s^2

1b) $t = 4$ and $t = 9.5$

1c) Toward the origin (moving right from $x = -3$)

1d) $\dfrac{5}{\pi}$

2a) 7.25 shirts/hour^2 (rate production rate is increasing)

2b) 268 shirts; underestimate (concave up)

2c) $1.76723 \le t \le 5.88617$

2d) -24.066 shirts/hour (selling faster than producing)

3a) -4 m/s^2

3b) Slowing down ($v > 0$, $a < 0$)

3c) 12 m/s

3d) $\sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{A\frac{dB}{dt} - B\frac{dA}{dt}}{A^2}$

3e) Angle of elevation decreasing at 0.0073 rad/s