Name: ______________________ Mark: ______________________

GCD and LCM

GCD stands for *greatest common divisor*, denoted $\gcd(a, b)$. Its name tells you exactly what to look for:

(1) Consider the (positive) divisors of a and the divisors of b.

(2) Restrict to just the divisors common to both lists found in (1).

(3) Get the greatest value from (2).

For small numbers, this is easy enough to check with brute force, since we can consider the largest divisors and check them first.

Example 1. Find $\gcd(12, 18)$

Solution 1. Clearly 6 is a divisor of both. This is the largest common divisor since 12 is the only divisor of 12 larger than 6, but 12 is not a factor of 18. Thus $\gcd(12, 18) = 6$.

Sometimes, it is faster to get the gcd of two numbers by making use of their prime factorization.

(1) To be a divisor of a number, each prime must use a power that is less than or equal to the power present in the prime factorization.

(2) Then to be a common divisor to both numbers, the power of each prime cannot be larger than the smaller power of that prime in each factorization.

(3) Finally, to be the greatest value possible, simply use that smallest power.

Thus, an algorithm for finding gcd boils down to:

(1) Get the prime factorization of a and b.

(2) For each prime, use the smaller power that appears in a and b.

Example 2. Find $\gcd(4620, 1694)$.

Solution 2. We can employ our prime factorization techniques to get

$$4620 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$
$$1694 = 2 \cdot 7 \cdot 11^2$$

Then

$$\gcd(4620, 1694) = 2 \cdot 7 \cdot 11 = 154$$

LCM stands for *least common multiple*, denoted $\operatorname{lcm}(a, b)$. Its name tells you exactly what to look for:

(1) Consider the (positive) multiples of a and the multiples of b.

(2) Restrict to just the multiples common to both lists found in (1).

(3) Get the least value from (2).

For small numbers, this is easy enough to check with brute force, since we can consider the smallest multiples and check them first.

Example 3. Find $\text{lcm}(12, 18)$

Solution 3. 18 is not a multiple of 12, but 36 is, so $\text{lcm}(12, 18) = 36$.

Sometimes, it is faster to get the lcm of two numbers by making use of their prime factorization.

(1) To be a multiple of a number, each prime must use a power that is greater than or equal to the power present in the prime factorization.

(2) Then to be a common multiple to both numbers, the power of each prime cannot be smaller than the larger power of that prime in each factorization.

(3) Finally, to be the least value possible, simply use that largest power.

Thus, an algorithm for finding lcm boils down to:

(1) Get the prime factorization of a and b.

(2) For each prime, use the larger power that appears in a and b.

Example 4. Find $\gcd(4620, 1694)$.

Solution 4. As before, we get

$$4620 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$
$$1694 = 2 \cdot 7 \cdot 11 \cdot 11$$

Then

$$\text{lcm}(4620, 1694) = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^2 = 50820$$

For large numbers like this (and especially when using the prime factorization technique), it is fairly standard to leave the answer as a prime factorization in index form.

Note: from the prime factorization algorithm for finding gcd and lcm, it is immediately clear that

$$ab = \gcd(a, b) \cdot \text{lcm}(a, b)$$

Finally, we note that the notions of gcd and lcm can extend to more than two numbers, with similar definitions and algorithms.

1. $\gcd(12, 18)$

2. $\gcd(24, 36)$

3. $\gcd(48, 64)$

4. $\gcd(27, 45)$

5. $\operatorname{lcm}(8, 12)$

6. $\operatorname{lcm}(15, 20)$

7. $\operatorname{lcm}(14, 21)$

8. $\operatorname{lcm}(9, 12)$

9. $\gcd(840, 990)$

10. $\operatorname{lcm}(1050, 495)$

11. $\gcd(1248, 1716)$

12. $\operatorname{lcm}(1632, 2280)$

13. $\gcd(12, 18, 24)$

14. $\gcd(15, 25, 35)$

15. $\gcd(20, 30, 40)$

16. $\gcd(16, 24, 36)$

17. $\operatorname{lcm}(4, 6, 9)$

18. $\operatorname{lcm}(3, 5, 7)$

19. $\operatorname{lcm}(5, 8, 10)$

20. $\operatorname{lcm}(6, 10, 15)$

21. $\gcd(840, 1260, 1890)$

22. $\operatorname{lcm}(360, 540, 810)$

23. $\gcd(1176, 1764, 2646)$

24. $\operatorname{lcm}(792, 936, 1144)$