Mini-math Div 3/4: Wednesday, January 22, 2025 (9.6-9.9) - (24 minutes)

Calculator active

SOLUTIONS

1. (4 points) The velocity vector of a particle moving in the plane is given by

$$\langle 5 - 2\cos(t^2), 8\sin(t^2)\cos(e^t)\rangle, \text{ for } 0 \le t \le 2$$

At time $t = 0$, the particle is at position $(3, -1)$. Write an equation for the line tangent to the path of the particle at $t = 1$.

Solution:

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{y'(1)}{x'(1)} = -1.566$$

$$x(1) = 3 + \int_0^1 x'(t)\,dt = 6.191$$

$$y(1) = -1 + \int_0^1 y'(t)\,dt = -2.241$$

so an equation is given by

$$y + 2.241 = -1.566(x - 6.191)$$

2. (4 points) Where does the graph $r = 1 - \sin\theta$, $0 \le \theta \le 2\pi$, have a vertical tangent?

Solution: To have a vertical tangent, we want $x'(\theta) = 0$ (and $y'(\theta) \neq 0$). We get

$$0 = x'(\theta) = \frac{d}{d\theta}(r\cos\theta) = \frac{d}{d\theta}(\cos\theta - \sin\theta\cos\theta) = -\sin\theta - \cos^2\theta + \sin^2\theta$$

With a calculator, $\theta = \pi/2, 7\pi/6, 11\pi/6$ (or $1.571, 3.665, 5.760$). However, we check

$$0 = y'(\theta) = \frac{d}{d\theta}(r\sin\theta) = \frac{d}{d\theta}(\sin\theta - \sin^2\theta) = \cos\theta - 2\sin\theta\cos\theta$$

Checking our three values, $y'(\pi/2) = 0$, but the other two yield non-zero y'. Therefore, the answer is $\theta = 7\pi/6, 11\pi/6$ (or $3.665, 5.760$).

3. (4 points) Find the area of the inner loop of $r = 4\sqrt{3} - 8\cos\theta$

Solution: The point of intersection with the origin is found by solving:

$$\begin{aligned} 0 &= 4\sqrt{3} - 8\cos\theta \\ \cos\theta &= \frac{\sqrt{3}}{2} \\ \theta &= \frac{\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k \end{aligned}$$

Since we want the integration bound to be increasing, we use $\frac{11\pi}{6} - 2\pi = -\frac{\pi}{6}$ and calculate

$$\int_{-\pi/6}^{\pi/6} \frac{1}{2}[\sqrt{3} - 2\cos\theta]^2 \, d\theta \approx 0.319$$

4. (4 points) Find the area of the region common to $r = 1 - \sin\theta$ and $r = 2\sin\theta$.

Solution: The graphs intersect at

$$\begin{aligned} 1 - \sin\theta &= 2\sin\theta \\ \theta &\approx 0.33984, 2.80176 \end{aligned}$$

With a graph, we see that $1-\sin\theta$ is the desired bounding function only on $[0.33984, 2.80176]$, while $2\sin\theta$ is the desired bounding function on $[0, 0.33984] \cup [2.80176, \pi]$. Then the area is

$$\frac{1}{2}\int_0^{0.33984} (2\sin\theta)^2 \, d\theta + \frac{1}{2}\int_{0.33984}^{2.80176} (1 - \sin\theta)^2 \, d\theta + \frac{1}{2}\int_{2.80176}^{\pi} (2\sin\theta)^2 \, d\theta \approx 0.169$$

Alternatively, we can use symmetry and get the area as

$$2\left(\frac{1}{2}\int_0^{0.33984} (2\sin\theta)^2 \, d\theta + \frac{1}{2}\int_{0.33984}^{\pi/2} (1 - \sin\theta)^2 \, d\theta\right) \approx 0.169$$