# Renert School: Integration Bee 2024–2025

**Problem 1.**

$$\int_1^{mower} \frac{1}{u}\,du$$

**Solution:**

$$\int_1^{mower} \frac{1}{u}\,du = \ln|u|\Big|_1^{mower} = \ln|mower|$$

Problems 2-17 are great review.

**Problem 2.**

$$\int_{-8}^{-1} \left(\frac{2}{x^{1/3}} + 4x\right)\,dx$$

**Solution:**

$$\int_{-8}^{-1} \left(\frac{2}{x^{1/3}} + 4x\right)\,dx = \left(3x^{2/3} + 2x^2\right)\Big|_{-8}^{-1} = 3((-1)^{2/3} - (-8)^{2/3}) + 2((-1)^2 - (-8)^2)$$
$$= 3(1-4) + 2(1-64) = -9 - 126 = -135$$

**Problem 3.**

$$\int \frac{\sqrt{r} - 5}{\sqrt{r}}\,dr$$

**Solution:** Divide to get

$$\int \frac{\sqrt{r} - 5}{\sqrt{r}}\,dr = \int (1 - 5r^{-1/2})\,dr = r - 10\sqrt{r} + C$$

**Problem 4.**

$$\int s(\sqrt{s} + 1)\,ds$$

**Solution:** Expand to get

$$\int s(\sqrt{s} + 1)\,ds = \int (s^{3/2} + s)\,ds = \frac{2}{5}s^{5/2} + \frac{1}{2}s^2 + C$$

**Problem 5.**

$$\int_{\pi/6}^{\pi/3} \sin(2x)\,dx$$

**Solution:** Method 1: Using $u = 2x$, $du = 2\,dx$, $\pi/6 \mapsto \pi/3$, $\pi/3 \mapsto 2\pi/3$,

$$\int_{\pi/6}^{\pi/3} \sin(2x)\,dx = \int_{\pi/3}^{2\pi/3} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos u\Big|_{\pi/3}^{2\pi/3} = -\frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}$$

Method 2: Integrating directly,

$$\int_{\pi/6}^{\pi/3} \sin(2x)\,dx = -\frac{1}{2}\cos(2x)\Big|_{\pi/6}^{\pi/3} = -\frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) = \frac{1}{2}$$

**Problem 6.**

$$\int x^3 e^{x^4}\,dx$$

**Solution:** Using $u = x^4$,

$$\int x^3 e^{x^4}\,dx = \int \frac{1}{4}e^u\,du = \frac{1}{4}e^{x^4} + C$$

**Problem 7.**

$$\int (2x-1)\sin(4x)\,dx$$

**Solution:** Use $f = 2x - 1$, $g' = \sin(4x)$, so $f' = 2$, $g = -\frac{1}{4}\cos(4x)$. By IBP,

$$\begin{aligned}\int (2x-1)\sin(4x)\,dx &= -\frac{2x-1}{4}\cos(4x) + \frac{1}{2}\int \cos(4x)\,dx \\ &= -\frac{2x-1}{4}\cos(4x) + \frac{1}{8}\sin(4x) + C\end{aligned}$$

**Problem 8.**

$$\int \frac{\sec\theta\tan\theta}{\sqrt{\sec\theta + 1}}\,d\theta$$

**Solution:** Using $u = \sec\theta + 1$, $du = \sec\theta\tan\theta\,d\theta$, so

$$\int \frac{\sec\theta\tan\theta}{\sqrt{\sec\theta+1}}\,d\theta = \int \frac{du}{\sqrt{u}} = 2u^{1/2} + C = 2\sqrt{\sec\theta + 1} + C$$

**Problem 9.**

$$\int \ln x^3\,dx$$

**Solution:** Method 1:

Use $f = \ln x^3, g' = 1$, so $f' = \frac{3x^2}{x^3} = \frac{3}{x}, g = x$. By IBP,

$$\int \ln x^3 \, dx = x \ln x^3 - \int 3 \, dx = x \ln x^3 - 3x + C$$

Method 2:

Simplify first, then

$$\int \ln x^3 \, dx = \int 3 \ln x \, dx = 3(x \ln x - x) + C$$

**Problem 10.**

$$\int \frac{1}{t^2 - 6t + 10} \, dt$$

**Solution:**

$$\int \frac{1}{t^2 - 6t + 10} \, dt = \int \frac{1}{(t-3)^2 + 1} \, dt = \arctan(t - 3) + C$$

**Problem 11.**

$$\int_{-1}^{1} \frac{1}{x^3} \, dx$$

**Solution:** This is an improper integral of type II - you cannot use FTC II or that this is an odd function over a symmetric interval. Instead, the discontinuity is at 0, so

$$\int_{-1}^{1} \frac{1}{x^3} \, dx = \int_{-1}^{0} \frac{1}{x^3} \, dx + \int_{0}^{1} \frac{1}{x^3} \, dx$$

Considering the second integral,

$$\int_{0}^{1} \frac{1}{x^3} \, dx = \lim_{c \to 0^+} \int_{c}^{1} \frac{1}{x^3} \, dx = \lim_{c \to 0^+} -\frac{1}{2x^2}\bigg|_c^1 = \lim_{c \to 0^+} \left( \frac{1}{2c^2} - \frac{1}{2} \right) = \infty$$

Therefore, the integral diverges.

**Problem 12.**

$$\int \frac{8x^2 - 2x + 3}{2x - 1} \, dx$$

**Solution:**

$$\begin{array}{r|l} & \quad 4x + 1 \\ 2x - 1 & \; 8x^2 - 2x + 3 \\ & -\,8x^2 + 4x \\ & \qquad 2x + 3 \\ & \quad -\,2x + 1 \\ & \qquad\quad 4 \end{array}$$

$$\int \frac{8x^2-2x+3}{2x-1}\,dx = \int \left(4x+1+\frac{4}{2x-1}\right)\,dx = 2x^2+x+2\ln|2x-1|+C$$

**Problem 13.**

$$\int x^2 e^{2x}\,dx$$

**Solution:** Use $f = x^2, g' = e^{2x}$, so $f' = 2x, g = \frac{1}{2}e^{2x}$. By IBP,

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$$

Use $f = x, g' = e^{2x}$, so $f' = 1, g = \frac{1}{2}e^{2x}$. By IBP again,

$$\int x^2 e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{2}\int e^{2x}\,dx = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C$$

**Problem 14.**

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx$$

**Solution:** Method 1: Using $u = \ln x$, $du = \frac{1}{x}\,dx$, $e \mapsto 1$, $e^2 \mapsto 2$, so

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_1^2 \frac{1}{u}\,du = \ln|u|\Big|_1^2 = \ln 2$$

Method 2: Integrating directly,

$$\int_e^{e^2} \frac{1}{x\ln x}\,dx = \ln\ln x\Big|_e^{e^2} = \ln\ln e^2 - \ln\ln e = \ln 2$$

**Problem 15.**

$$\int_0^4 \sqrt{3x+4}\,dx$$

**Solution:** Method 1: Using $u = 3x+4$, $du = 3\,dx$, $0 \mapsto 4$, $4 \mapsto 16$, so

$$\int_0^4 \sqrt{3x+4}\,dx = \int_4^{16} \frac{1}{3}\sqrt{u}\,du = \frac{2}{9}u^{3/2}\Big|_4^{16} = \frac{2}{9}(16^{3/2} - 4^{3/2}) = \frac{2}{9}(4^3 - 2^3) = \frac{112}{9}$$

Method 2: Integrating directly,

$$\int_0^4 \sqrt{3x+4}\,dx = \frac{2}{9}\sqrt{3x+4}\Big|_0^4 = \frac{2}{9}(16^{3/2} - 4^{3/2}) = \frac{2}{9}(4^3 - 2^3) = \frac{112}{9}$$

**Problem 16.**

$$\int \frac{x+1}{x(2x+1)}\,dx$$

**Solution:** If $\dfrac{x+1}{x(2x+1)} = \dfrac{A}{x} + \dfrac{B}{2x+1}$, the Heaviside cover-up method gives $A = \frac{0+1}{2(0)+1} = 1$ and $B = \frac{-\frac{1}{2}+1}{-\frac{1}{2}} = -1$, so

$$\int \frac{x+1}{x(2x+1)}\,dx = \int \left(\frac{1}{x} + \frac{-1}{2x+1}\right)\,dx = \ln|x| - \frac{1}{2}\ln|2x+1| + C$$

**Problem 17.**

$$\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$$

**Solution:** The upper limit is a discontinuity, so

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{2-x}}\,dx &= \lim_{b\to 2^-}(-2\sqrt{2-x})\Big|_0^b \\ &= \lim_{b\to 2^-}(-2\sqrt{2-b} + 2\sqrt{2-0})) = 2\sqrt{2}\end{aligned}$$