$\sqrt{2}$ is irrational

Suppose $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = \dfrac{a}{b}$ for $a, b \in \mathbb{Z}, b \neq 0$, with $\gcd(a, b) = 1$. Then

$$\begin{aligned} \sqrt{2}b &= a \\ 2b^2 &= a^2 \end{aligned}$$

Notice $b^2 \in \mathbb{Z}$ since $b \in zz$ and the integers are closed under multiplication, so $2 \mid a^2$. Since 2 is prime, this implies $2 \mid a$, so that $a = 2c$ for some integer c. Substituting,

$$\begin{aligned} 2b^2 &= (2c)^2 = 4c^2 \\ b^2 &= 2c^2 \end{aligned}$$

Again, $c^2 \in \mathbb{Z}$ since $c \in zz$ and the integers are closed under multiplication, so $2 \mid b^2$. Since 2 is prime, this implies $2 \mid b$ But then $2 \mid \gcd(a, b)$, a contradiction to $\gcd(a, b) = 1$. Therefore, $\sqrt{2} \notin \mathbb{Q}$.