Mini-math Div 3/4: Friday, January 12, 2024 (9.1-9.6) - 20 minutes

SOLUTIONS

1. (3 points) Write an equation for the line tangent to the curve defined by $r(t) = \langle 2^t, 1/t \rangle$ at the point where $x = 8$.

Solution: $2^t = 8$ gives $t = 3$. At this value, $y(3) = 1/3$. Now,

$$\frac{dy}{dx}\Big|_{t=3} = \frac{dy/dt}{dx/dt}\Big|_{t=3} = \frac{-t^{-2}}{2^t \ln 2}\Big|_{t=3} = \frac{-1/9}{8 \ln 2} = -\frac{1}{72 \ln 2}$$

by point-slope, an equation of the tangent line is

$$y - \frac{1}{3} = -\frac{1}{72 \ln 2}(x - 8)$$

2. (4 points) If $x(\theta) = \tan 2\theta$ and $y(\theta) = \sec 2\theta$, find the concavity at $\theta = \pi/6$.

Solution:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \sec 2\theta \tan 2\theta}{2 \sec^2 2\theta} = \sin 2\theta$$

Then

$$\frac{d^2y}{dx^2}\Big|_{\theta=\pi/6} = \frac{\frac{d}{d\theta}\left(\frac{dy/d\theta}{dx/d\theta}\right)}{dx/d\theta}\Big|_{\theta=\pi/6} = \frac{\frac{d}{d\theta}(\sin 2\theta)}{2 \sec^2 2\theta}\Big|_{\theta=\pi/6} = \frac{2 \cos 2\theta}{2 \sec^2 2\theta}\Big|_{\theta=\pi/6} = (\cos \pi/3)^3 = \frac{1}{8}$$

Concave up.

3. (2 points) Write down (but do not evaluate) an integral which represents the length of the curve described by the parametric equations $x = t^3/3$ and $y = t^2/2$ from $t = 0$ to $t = 1$. (Extra challenge: find the exact value.)

Solution: First, note that $x'(t) = t^2$ and $y'(t) = t$

$$L = \int_0^1 \sqrt{t^4 + t^2}\, dt \quad \left(= \frac{(t^2+1)^{3/2}}{3}\Big|_0^1 = \frac{2\sqrt{2} - 1}{3} \right)$$

4. (3 points) If f is a vector-valued function defined by $f(t) = \langle 2\sin t, \cos 2t\rangle$, then what is $f''(\pi/3)$?

Solution:

$$\begin{aligned} f'(t) &= \langle 2\cos t, -2\sin 2t\rangle, \\ f''(t) &= \langle -2\sin t, -4\cos 2t\rangle, \\ f''(\pi/3) &= \langle -\sqrt{3}, 2\rangle \end{aligned}$$

5. (3 points) Find the vector-valued function $f(t)$ that satisfies the initial conditions $f(1) = \langle 4, 5\rangle$, and $f'(t) = \langle 6t, 7\rangle$.

Solution:

$$\begin{aligned} f(t) &= \langle 4 + \int_1^t 6u\,du, 5 + \int_1^t 7\,du\rangle \\ &= \langle 4 + 3(t^2 - 1), 5 + 7(t - 1)\rangle \end{aligned}$$

6. (4 points) (Calculator-active) At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle \sin(t^2), 2^{\sqrt{t}}\rangle$. If the particle is at point $(-3, 1)$ at time $t = 0$, how far is the particle from the origin at time $t = 3$?

Solution: We calculate

$$\begin{aligned} x(3) &= -3 + \int_0^3 \sin(t^2)\,dt \approx -2.22644 \\ y(3) &= 1 + \int_0^3 2^{\sqrt{t}}\,dt \approx 7.93628 \end{aligned}$$

Then

$$d = \sqrt{[x(3)]^2 + [y(3)]^2} \approx 8.243$$