Mini-math Div 3/4: Monday, November 20, 2023 (15 minutes)

SOLUTIONS

1. (2 points) The graph of the piecewise linear function f is shown in the figure to the right. What is the average value of f over $[-3, 5]$?

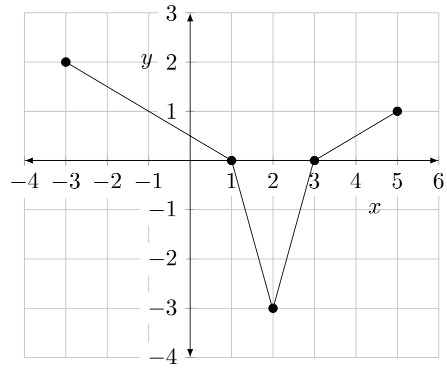
   A. -1

   B. $-1/8$

   C. 0

   D. $1/4$

   E. 2



Solution:

$$f_{avg} = \frac{\int_{-3}^{5} f\,dx}{5-(-3)} = \frac{4-3+1}{8} = \frac{1}{4}$$

D is correct.

2. (2 points) The graph of the velocity of a function is the piecewise linear function shown in the figure to the right. The initial position of the particle at time $t = 0$ is $x = 1$. What is the total distance the particle travels from $t = 0$ to $t = 6$?

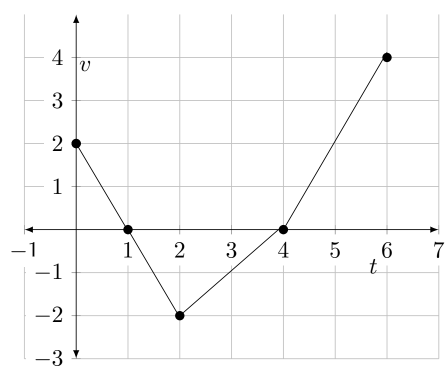
   A. 2

   B. 3

   C. 4

   D. 8

   E. 9



Solution:

$$\int_0^6 |v(t)|\,dt = 1 + 3 + 4 = 8$$

D is correct.

3. (2 points) The acceleration of a particle is modelled by $a(t) = 2t + 3$ for $t \geq 0$. At $t = 0$, the velocity of the particle is -2 and its position is 2.5. What is the change in displacement of the particle from $t = 0$ to $t = 3$?

A. 9 B. 16 **C. 16.5** D. 19 E. 22.5

Solution:

$$v(t) = \int a(t)\,dt = \int (2t+3)\,dt = t^2 + 3t + C$$

Since $v(0) = -2$, we know $C = -2$. Then the change in displacement is

$$\Delta x = \int_0^3 v(t)\,dt = \int_0^3 \left(t^2 + 3t - 2\right)\,dt = \left(\frac{1}{3}t^3 + \frac{3}{2}t^2 - 2t\right)\Big|_0^3 = 9 + \frac{27}{2} - 6 = 16.5$$

C is correct.

4. (2 points) Suppose f is a differentiable function. Which of the following statements are true:

(I) The average value of the derivative of f over $[a, b]$ is the same as the average rate of change of f over $[a, b]$.

(II) There exists a $c \in [a, b]$ for which $f(c)$ equals the average value of f over $[a, b]$.

A. (I) only B. (II) only **C. Both (I) and (II)** D. Neither (I) nor (II)

E. The truth of both statements depend on the specific choice of f

Solution: By FTC II,

$$\frac{\int_a^b f'(x)\,dx}{b-a} = \frac{f(b) - f(a)}{b-a}$$

so (I) is true.

Since f is continuous and $[a, b]$ is a closed and bounded interval, the Extreme Value Theorem tells us that there are m and M for which $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then

$$\frac{\int_a^b m\,dx}{b-a} \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq \frac{\int_a^b M\,dx}{b-a}$$
$$m \leq f_{avg} \leq M$$

By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $f(c) = f_{avg}$, so (II) is true.

C is correct.

5. (2 points) Water is leaking out of a tub at a rate modelled by $r(t) = \dfrac{1}{t^2+1}\text{cm}^3/\text{min}$, where t is in minutes. If the initial volume of the tub is 160 000 cm^3, which of the following represents the volume of the tub at time t?

A. $160000 + \displaystyle\int_0^t r(x)\,dx$

B. $160000 - \displaystyle\int_0^t r(x)\,dx$

C. $160000 - \dfrac{1}{t^2+1}$

D. $160000 + \dfrac{r(t)}{t^2+1}$

E. $\dfrac{1}{t^2+1}$

Solution: By FTC II,

$$-\int_0^t r(t)\,dt = V(t) - V(0)$$
$$V(t) = 160000 - \int_0^t r(t)\,dt$$

B is correct.

6. (2 points) Find the area of the bounded region in the first quadrant below both $y = x^2$ and $y = 2 - x$ and above the x-axis.

A. 2/3 **B. 5/6** C. 1 D. 7/6 E. 3

Solution: Integrating with respect to y,

$$A = \int_0^1 [(2-y) - \sqrt{y}]\,dy = \left(2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right)\Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$$

OR Integrating with respect to x (with 2 regions),

$$A = \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx = \frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{3} + 4 - 2 - 2 + \frac{1}{2} = \frac{5}{6}$$

B is correct.

7. (4 points) Write an integral (or integrals) to calculate the area of the finite region(s) bounded by the given curves.

$$x + y = 1, \quad 2x - y = -1, \quad 4x - y = 4$$

Solution: Finding the intersections of the curves:

$y = 1 - x$ and $y = 2x + 1$ intersect at $x = 0$

$y = 1 - x$ and $y = 4x - 4$ intersect at $x = 1$

$y = 2x + 1$ and $y = 4x - 4$ intersect at $x = 5/2$

On $[0, 1]$, $2x + 1 \geq 1 - x$ and on $[1, 5/2]$, $2x + 1 \geq 4x - 4$, so the area is

$$\begin{aligned}
&\int_0^1 [(2x+1) - (1-x)]\,dx + \int_1^{5/2} [(2x+1) - (4x-4)]\,dx \\
&= \int_0^1 3x\,dx + \int_1^{5/2} (5 - 2x)\,dx = \frac{3}{2}x^2\Big|_0^1 + (5x - x^2)\Big|_1^{5/2} \\
&= \frac{3}{2} + \left(\left(5 \cdot \frac{5}{2} - \frac{25}{4}\right) - (5 - 1)\right) \\
&= \frac{3}{2} + \frac{25}{4} - 4 = \frac{31 - 16}{4} = \frac{15}{4}
\end{aligned}$$

OR

In terms of y,

$$\int_0^1 [(y+4)/4 - (1-y)]\,dy + \int_1^6 [(y+4)/4 - (y-1)/2]\,dy = \frac{5}{8} + \frac{25}{8} = \frac{15}{4}$$