Mini-math Div 3/4: Monday, October 23, 2023 (20 minutes)

SOLUTIONS

1. (2 points) Solve the following differential equation:

$$\frac{dy}{dx} = xy\sin(x^2) \cdot \ln y$$

Solution:

$$\begin{aligned}
\int \frac{1}{y\ln y}\,dy &= \int x\sin x^2\,dx \\
\ln|\ln y| &= -\frac{1}{2}\cos x^2 + C_1 \\
|\ln y| &= e^{-\frac{1}{2}\cos x^2 + C_1} \\
\ln y &= \pm e^{-\frac{1}{2}\cos x^2 + C_1} \quad \text{or} \quad C_2 e^{-\frac{1}{2}\cos x^2} \\
y &= e^{\pm e^{-\frac{1}{2}\cos x^2 + C_1}} \quad \text{or} \quad e^{C_2 e^{-\frac{1}{2}\cos x^2}}
\end{aligned}$$

2. (2 points) During a chemical reaction, the rate of change of the amount of the chemical remaining is proportional to the amount remaining. At time $t = 0$, the amount of the chemical is 60 g. At time $t = 8$, the amount of the chemical is 12 g. At what time t is the amount of the chemical 4 g?

A. $\frac{4\sqrt{42}}{3}$ B. $\frac{28}{3}$ **C.** $\frac{8\ln 15}{\ln 5}$ D. $\frac{8\ln 4}{\ln 12}$ E. $\frac{8}{3}$

Solution:

$$P(t) = P_0 e^{kt} = P_0 (e^k)^t$$

$$P_1 = P_0 (e^k)^{t_1} \quad \Rightarrow \quad e^k = \left(\frac{P_1}{P_0}\right)^{1/t_1}$$

$$P(t) = P_0 \left(\frac{P_1}{P_0}\right)^{t/t_1}$$

$$P_2 = P_0 \left(\frac{P_1}{P_0}\right)^{t_2/t_1} \quad \Rightarrow \quad t_2 = \frac{t_1 \ln(P_2/P_0)}{\ln(P_1/P_0)} = \frac{8\ln(1/15)}{\ln(1/5)} = \frac{8\ln 15}{\ln 5}$$

(C)

3. (2 points) Solve the following initial value problem:

$$\frac{dy}{dx} = y^2, \quad y(3) = -2$$

A. $y = \dfrac{1}{5/2 - x}$ for $x \neq 5/2$

B. $y = \dfrac{2}{5 - 2x}$ for $x > 5/2$

C. $y = -\dfrac{1}{x} - \dfrac{5}{3}$ for $x \neq 0$

D. $y = -\dfrac{5x + 3}{3x}$ for $x > 0$

Solution:

$$\int \frac{1}{y^2}\,dy = \int 1\,dx \quad \Rightarrow \quad -\frac{1}{y} = x + C,$$
$$\frac{1}{2} = 3 + C \quad \Rightarrow C = -\frac{5}{2},$$
$$y = \frac{1}{5/2 - x} = \frac{2}{5 - 2x}$$

Since the domain is the largest open interval which contains the initial condition, the domain is $x > 5/2$.

(B)

4. The number of squirrels in a park at time t is modelled by the function $y = F(t)$ that satisfies the logistic differential equation $\frac{dy}{dt} = \frac{1}{2000}y(1500 - y)$, where $t \geq 0$ is measured in weeks. The number of squirrels in the park at time $t = 0$ is $F(0) = b$, where b is a positive constant.

(a) i. (1 point) If $b = 300$, what is the largest rate of increase in the number of squirrels in the park?

Solution: Since $300 < 1500/2 = 750$, F grows most rapidly when it is half the carrying capacity, 750. Then

$$\left.\frac{dy}{dt}\right|_{F=750} = \frac{750}{2000}(1500 - 750) = \frac{1125}{4} = 281.25$$

ii. (1 point) If $b = 1000$, what is the largest rate of increase in the number of squirrels in the park?

Solution: For $F > 750$, $\frac{dy}{dt} > 0$ and $\frac{d^2y}{dt^2} < 0$, so F grows most rapidly when $F = 1000$.

$$\left.\frac{dy}{dt}\right|_{F=1000} = \frac{1000}{2000}(1500 - 1000) = 250$$

(b) (2 points) If $b = 150$, find $\lim_{t\to\infty} F(t)$ and interpret the meaning of this limit in the context of the problem. For reference, the differential equation is $\frac{dy}{dt} = \frac{1}{2000}y(1500 - y)$.

Solution: The carrying capacity is 1500, so $\lim_{t\to\infty} F(t) = 1500$.
This means in the long term, the population of squirrels in the park will tend to 1500.

(c) (4 points) (*) Find the function $F(t)$ if $b = 500$.

Solution:

$$\begin{aligned}
\int \frac{dy}{y(1500-y)} &= \int \frac{dt}{2000} \\
\int \left(\frac{1/1500}{y} + \frac{1/1500}{1500-y}\right) &= \int \frac{dt}{2000} \\
\ln\left|\frac{y}{1500-y}\right| = (\ln|y| - \ln|1500-y|) &= \frac{1500}{2000}x + C = \frac{3}{4}x + C \\
\frac{y}{1500-y} &= Ce^{3t/4}
\end{aligned}$$

Using the initial condition,

$$\frac{1}{2} = \frac{500}{1500-500} = C$$

so

$$\begin{aligned}
\frac{2y}{1500-y} &= e^{3t/4} \\
2y &= 1500e^{3t/4} - e^{3t/4}y \\
y &= \frac{1500e^{3t/4}}{e^{3t/4}+2}
\end{aligned}$$

(Alternatively, go directly to the general solution if you have memorized it. Be careful with using the correct general solution which depends on the form of the logistic DE.)