Renert School: Series Bee 2023–2024

Problem 1. If the nth partial sum of a series $\sum_{k=1}^{\infty} a_k$ is

$$S_n = \frac{3n+2}{2n-3}$$

find a_4.

Solution:

$$\begin{aligned} a_4 &= S_4 - S_3 \\ &= \frac{3(4)+2}{2(4)-3} - \frac{3(3)+2}{2(3)-3} \\ &= \frac{14}{5} - \frac{11}{3} \\ &= \frac{42-55}{15} = -\frac{13}{15} \end{aligned}$$

Problem 2. If the Nth partial sum of a series $\sum_{n=2}^{\infty} a_n$ is

$$S_N = \frac{(\ln N)^2}{N}$$

find $\displaystyle\sum_{n=2}^{\infty} a_n$.

Solution:

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{(\ln N)^2}{N} = 0$$

Problem 3. Find the sum of the infinite series

$$5 - \frac{5}{3} + \frac{5}{9} - \frac{5}{27} + \cdots$$

Solution: This is a geometric series with $a = 5$ and $r = -5/3$:

$$S = \frac{5}{1-(-1/3)} = \frac{15}{3+1} = \frac{15}{4}$$

Problem 4. Express $2.0\overline{63}$ as a ratio of two integers in lowest terms.

Solution:

$$\begin{aligned}
2.0\overline{63} &= 2 + 0.063 + 0.00063 + 0.0000063 + \cdots \\
&= 2 + \frac{63}{10^3} + \frac{63}{10^5} + \frac{63}{10^7} + \cdots \\
&= 2 + \frac{63/1000}{1-(1/100)} = 2 + \frac{63}{1000-10} \\
&= 2 + \frac{63}{990} = 2 + \frac{7}{110} \\
&= \frac{220+7}{110} = \frac{227}{110}
\end{aligned}$$

Problem 5. Is the series $S = \sum_{n=1}^{\infty} 3^{3n} 5^{2-2n}$ convergent or divergent? If convergent, find its sum.

Solution: This is a geometric series with $r = 3^3/5^2 = 27/25 > 1$, so is divergent. You could also use the Divergence Test instead.

Problem 6. Find the sum of the series

$$\sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

Solution: The Nth partial sum is

$$\begin{aligned}
S_N &= \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+2} \right) \\
&= \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}} \right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6} \right) + \cdots + \left(\cancel{\frac{1}{N-1}} - \frac{1}{N+1} \right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+2} \right) \\
&= \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2}
\end{aligned}$$

Then

$$\sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right) = \lim_{N\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} \right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Problem 7. If $\sum_{n=1}^{\infty} a_n = 2$ and $\sum_{n=1}^{\infty} b_n = -3$, find the sum of the infinite series $\sum_{n=1}^{\infty} (5a_n - 2b_n)$

Solution: Since both are convergent series,

$$\sum_{n=1}^{\infty}(5a_n - 2b_n) = 5\sum_{n=1}^{\infty} a_n - 2\sum_{n=1}^{\infty} b_n = 5(2) - 2(-3) = 16$$

Problem 8. Find the value of p so that the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ is convergent.

Solution: If $p \leq 0$, the series is clearly divergent by comparison to the Harmonic Series. If $p > 0$, then $\frac{1}{x(\ln x)^p}$ is a positive, decreasing, continuous function on $[2, \infty)$. We check

$$\begin{aligned}\int_2^{\infty}\frac{1}{x(\ln x)^p}\,dx &= \lim_{b\to\infty}\int_2^{b}\frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty}\frac{1}{1-p}(\ln x)^{1-p}\Big|_2^b \\ &= \lim_{n\to\infty}\frac{1}{1-p}((\ln b)^{1-p} - (\ln 2)^{1-p})\end{aligned}$$

This limit is finite if and only if $1 - p < 0$, that is, if $p > 1$.

Problem 9. Does the following series converge or diverge, and why?

$$\sum_{n=2}^{\infty}\frac{3}{n^2 - 6n}$$

Solution: Converges using Limit Comparison Test with $b_n = \frac{1}{n^2}$.

Problem 10. Does the following series converge or diverge, and why?

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$$

Solution: Clearly, $\frac{1}{\ln n}$ is decreasing to 0 and the series is alternating, so the series converges by the Alternating Series Test.

Problem 11. Does the following series converge or diverge, and why?

$$\sum \frac{2^{n+1} - 1}{2^{n+3}}$$

Solution:

$$\lim_{n\to\infty}\frac{2^{n+1} - 1}{2^{n+3}} = \lim_{n\to\infty}\frac{2 - \frac{1}{2^n}}{2^3} = \frac{2}{8} \neq 0$$

so the series diverges by the Divergence Test,

Problem 12. Does the following series converge or diverge, and why?

$$\sum_{n=1}^{\infty} n^2 e^{-2n}$$

Solution:

$$\lim_{n\to\infty} \left| \frac{(n+1)^2 e^{-2(n+1)}}{n^2 e^{-2n}} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^2 \cdot e^{-2} = \frac{1}{e^2} < 1$$

so by the Ratio Test, the original series converges.

Note: Integral test also works, but you need to apply Integration by Parts twice

Problem 13. Does the following series converge or diverge, and why?

$$\sum_{n=1}^{\infty} \frac{3 + \sin n}{n^2}$$

Solution:

$$0 \leq \frac{3 + \sin n}{n^2} \leq \frac{4}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges by p series, the original series converges by Comparison Test.

Problem 14. Does the following series converge or diverge, and why?

$$\sum_{n=1}^{\infty} \frac{3 + \sin n}{n}$$

Solution:

$$0 \leq \frac{2}{n} \leq \frac{3 + \sin n}{n}$$

Since $\sum \frac{1}{n}$ diverges by p series, the original series diverges by Comparison Test.

Problem 15. Does the following series converge or diverge, and why?

$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2} \right)$$

Solution:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{m\to 0} \frac{\sin m}{m} = 1$$

Since $\sum \frac{1}{n^2}$ converges by p series, the original series converges by Limit Comparison Test.

Problem 16. Does the following series converge or diverge, and why?

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$$

Solution:

$$\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = 1$$

so the original series diverges by the Divergence Test.

Problem 17. Does the following series converge or diverge, and why?

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Solution:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| &= \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n+1)!}{n!} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n \cdot (n+1) \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = 1/e
\end{aligned}$$

where in the last limit, we used one of the definitions of e.

(Alternatively, we could use l'Hôpital's Rule combined with logarithms. If $L = \lim_{n\to\infty}(1 + \frac{1}{n})^{-n}$, then continuity gives

$$\begin{aligned}
\ln L &= \lim_{n\to\infty} -n \ln\left(1 + \frac{1}{n}\right) \\
&= \lim_{n\to\infty} -\frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \\
&= \lim_{x\to 0^+} -\frac{\ln(1+x)}{x} \\
&= \lim_{x\to 0^+} -\frac{\frac{1}{1+x}}{1} = -1
\end{aligned}$$

This means $L = e^{-1} = 1/e$.)

Since $1/e < 1$, the Ratio Test tells us that the original series converges.

Problem 18. Does the following series converge or diverge, and why?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{1/n}}{\sqrt{n}}$$

Solution: This is clearly an alternating series. If $b_n = \frac{e^{1/n}}{\sqrt{n}}$, then the numerator is decreasing while the denominator is increasing, so b_n is decreasing to 0, and the Alternating Series Test implies the original series is convergent.

Problem 19. Does the following series converge absolutely, converge conditionally, or diverge, and why?

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{\cos(\pi n)}{n}$$

Solution: The sum simplifies as

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{\cos(\pi n)}{n} = \sum_{n=1}^{\infty}(-1)^{n}\frac{(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges by p series.

Problem 20. Does the following series converge absolutely, converge conditionally, or diverge, and why?

$$\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \cdots$$

where in the alternating sum, the numerators are increasing by 1 and the denominators are increasing by the next odd number.

Solution: The series can be written as

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n^2+1}$$

so converges by the Alternating Series Test. However, the series of the absolute values diverges upon Limit Comparison Test with the Harmonic Series, hence the original series is conditionally convergent.

Problem 21. For what value(s) of x, if any, will the following series conditionally converge?

$$\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n3^n}$$

Solution:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(-1)^n x^n}\right| = \lim_{n\to\infty}\frac{x}{3}$$

By the Ratio Test, the series converges absolutely if $|x/3| < 1$ and diverges if $|x/3| > 1$. Then in order to converge conditionally, we need $|x/3| = 1$. Notice this does not guarantee conditional convergence on its own - we need to check convergence.

At $x = -3$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n(-3)^n}{n3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

which diverges by p series. At $x = 3$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{n3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

which converges by Alternating Series Test. Therefore, only at $x = 3$ do we get conditional convergence.