

Renert School: Integration Bee 2023–2024

Problem 1.

$$\int_1^{mower} \frac{1}{u} du$$

Solution:

$$\int_1^{mower} \frac{1}{u} du = \ln |u| \Big|_1^{mower} = \ln |mower|$$

Problems 2-17 are great review.

Problem 2.

$$\int_{-8}^{-1} \left(\frac{2}{x^{1/3}} + 4x \right) dx$$

Solution:

$$\begin{aligned} \int_{-8}^{-1} \left(\frac{2}{x^{1/3}} + 4x \right) dx &= (3x^{2/3} + 2x^2) \Big|_{-8}^{-1} = 3((-1)^{2/3} - (-8)^{2/3}) + 2((-1)^2 - (-8)^2) \\ &= 3(1 - 4) + 2(1 - 64) = -9 - 126 = -135 \end{aligned}$$

Problem 3.

$$\int \frac{\sqrt{r} - 5}{\sqrt{r}} dr$$

Solution: Divide to get

$$\int \frac{\sqrt{r} - 5}{\sqrt{r}} dr = \int (1 - 5r^{-1/2}) dr = r - 10\sqrt{r} + C$$

Problem 4.

$$\int s(\sqrt{s} + 1) ds$$

Solution: Expand to get

$$\int s(\sqrt{s} + 1) ds = \int (s^{3/2} + s) ds = \frac{2}{5}s^{5/2} + \frac{1}{2}s^2 + C$$

Problem 5.

$$\int_{\pi/6}^{\pi/3} \sin(2x) dx$$

Solution: Method 1: Using $u = 2x$, $du = 2 dx$, $\pi/6 \mapsto \pi/3$, $\pi/3 \mapsto 2\pi/3$,

$$\int_{\pi/6}^{\pi/3} \sin(2x) dx = \int_{\pi/3}^{2\pi/3} \frac{1}{2} \sin u du = -\frac{1}{2} \cos u \Big|_{\pi/3}^{2\pi/3} = -\frac{1}{2} \left(-\frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2}$$

Method 2: Integrating directly,

$$\int_{\pi/6}^{\pi/3} \sin(2x) dx = -\frac{1}{2} \cos(2x) \Big|_{\pi/6}^{\pi/3} = -\frac{1}{2} \left(-\frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2}$$

Problem 6.

$$\int x^3 e^{x^4} dx$$

Solution: Using $u = x^4$,

$$\int x^3 e^{x^4} dx = \int \frac{1}{4} e^u du = \frac{1}{4} e^{x^4} + C$$

Problem 7.

$$\int (2x - 1) \sin(4x) dx$$

Solution: Use $f = 2x - 1$, $g' = \sin(4x)$, so $f' = 2$, $g = -\frac{1}{4} \cos(4x)$. By IBP,

$$\begin{aligned} \int (2x - 1) \sin(4x) dx &= -\frac{2x - 1}{4} \cos(4x) + \frac{1}{2} \int \cos(4x) dx \\ &= -\frac{2x - 1}{4} \cos(4x) + \frac{1}{8} \sin(4x) + C \end{aligned}$$

Problem 8.

$$\int \frac{\sec \theta \tan \theta}{\sqrt{\sec \theta + 1}} d\theta$$

Solution: Using $u = \sec \theta + 1$, $du = \sec \theta \tan \theta d\theta$, so

$$\int \frac{\sec \theta \tan \theta}{\sqrt{\sec \theta + 1}} d\theta = \int \frac{du}{\sqrt{u}} = 2u^{1/2} + C = 2\sqrt{\sec x + 1} + C$$

Problem 9.

$$\int \ln x^3 dx$$

Solution: Method 1:

Use $f = \ln x^3, g' = 1$, so $f' = \frac{3x^2}{x^3} = \frac{3}{x}, g = x$. By IBP,

$$\int \ln x^3 dx = x \ln x^3 - \int 3 dx = x \ln x^3 - 3x + C$$

Method 2:

Simplify first, then

$$\int \ln x^3 dx = \int 3 \ln x dx = 3(x \ln x - x) + C$$

Problem 10.

$$\int \frac{1}{t^2 - 6t + 10} dt$$

Solution:

$$\int \frac{1}{t^2 - 6t + 10} dt = \int \frac{1}{(t - 3)^2 + 1} dt = \arctan(t - 3) + C$$

Problem 11.

$$\int_{-1}^1 \frac{1}{x^3} dx$$

Solution: This is an improper integral of type II - you cannot use FTC II or that this is an odd function over a symmetric interval. Instead, the discontinuity is at 0, so

$$\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx$$

Considering the second integral,

$$\int_0^1 \frac{1}{x^3} dx = \lim_{c \rightarrow 0^+} \int_c^1 \frac{1}{x^3} dx = \lim_{c \rightarrow 0^+} \left. -\frac{1}{2x^2} \right|_c^1 = \lim_{c \rightarrow 0^+} \left(\frac{1}{2c^2} - \frac{1}{2} \right) = \infty$$

Therefore, the integral diverges.

Problem 12.

$$\int \frac{8x^2 - 2x + 3}{2x - 1} dx$$

Solution:

$$\begin{array}{r} 4x + 1 \\ 2x - 1 \overline{) 8x^2 - 2x + 3} \\ \underline{- 8x^2 + 4x} \\ 2x + 3 \\ \underline{- 2x + 1} \\ 4 \end{array}$$

$$\int \frac{8x^2 - 2x + 3}{2x - 1} dx = \int \left(4x + 1 + \frac{4}{2x - 1} \right) dx = 2x^2 + x + 2 \ln |2x - 1| + C$$

Problem 13.

$$\int x^2 e^{2x} dx$$

Solution: Use $f = x^2, g' = e^{2x}$, so $f' = 2x, g = \frac{1}{2}e^{2x}$. By IBP,

$$\int x^2 e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \int x e^{2x} dx$$

Use $f = x, g' = e^{2x}$, so $f' = 1, g = \frac{1}{2}e^{2x}$. By IBP again,

$$\int x^2 e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{2} \int e^{2x} dx = \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} + \frac{1}{4} e^{2x} + C$$

Problem 14.

$$\int_e^{e^2} \frac{1}{x \ln x} dx$$

Solution: Method 1: Using $u = \ln x, du = \frac{1}{x} dx, e \mapsto 1, e^2 \mapsto 2$, so

$$\int_e^{e^2} \frac{1}{x \ln x} dx = \int_1^2 \frac{1}{u} du = \ln |u| \Big|_1^2 = \ln 2$$

Method 2: Integrating directly,

$$\int_e^{e^2} \frac{1}{x \ln x} dx = \ln \ln x \Big|_e^{e^2} = \ln \ln e^2 - \ln \ln e = \ln 2$$

Problem 15.

$$\int_0^4 \sqrt{3x + 4} dx$$

Solution: Method 1: Using $u = 3x + 4, du = 3 dx, 0 \mapsto 4, 4 \mapsto 16$, so

$$\int_0^4 \sqrt{3x + 4} dx = \int_4^{16} \frac{1}{3} \sqrt{u} du = \frac{2}{9} u^{3/2} \Big|_4^{16} = \frac{2}{9} (16^{3/2} - 4^{3/2}) = \frac{2}{9} (4^3 - 2^3) = \frac{112}{9}$$

Method 2: Integrating directly,

$$\int_0^4 \sqrt{3x + 4} dx = \frac{2}{9} \sqrt{3x + 4} \Big|_0^4 = \frac{2}{9} (16^{3/2} - 4^{3/2}) = \frac{2}{9} (4^3 - 2^3) = \frac{112}{9}$$

Problem 16.

$$\int \frac{x+1}{x(2x+1)} dx$$

Solution: If $\frac{x+1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$, the Heaviside cover-up method gives $A = \frac{0+1}{2(0)+1} = 1$ and $B = \frac{-\frac{1}{2}+1}{-\frac{1}{2}} = -1$, so

$$\int \frac{x+1}{x(2x+1)} dx = \int \left(\frac{1}{x} + \frac{-1}{2x+1} \right) dx = \ln|x| - \frac{1}{2} \ln|2x+1| + C$$

Problem 17.

$$\int_0^2 \frac{1}{\sqrt{2-x}} dx$$

Solution: The upper limit is a discontinuity, so

$$\begin{aligned} \int_0^2 \frac{1}{\sqrt{2-x}} dx &= \lim_{b \rightarrow 2^-} \int_0^b \frac{1}{\sqrt{2-x}} dx = \lim_{b \rightarrow 2^-} (-2\sqrt{2-x}) \Big|_0^b \\ &= \lim_{b \rightarrow 2^-} (-2\sqrt{2-b} + 2\sqrt{2-0}) = 2\sqrt{2} \end{aligned}$$