Proofs of Trigonometric Derivatives

The text on the right is not needed in the proof, but is included here to add to clarity.

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h-\sin x}{h}+\lim_{h\to 0}\frac{\sin h\cos x}{h} && \text{by the sine summation formula} \\
&= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h} && \text{since } \sin x \text{ and } \cos x \text{ are constants with respect to } h \\
&= \sin x\cdot 0+\cos x\cdot 1 && \text{by the first and second Fundamental Trig Limits} \\
&= \cos x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin h\sin x-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\cos x}{h}-\lim_{h\to 0}\frac{\sin h\sin x}{h} && \text{by the cosine summation formula} \\
&= \cos x\lim_{h\to 0}\frac{\cos h-1}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h} && \text{since } \sin x \text{ and } \cos x \text{ are constants with respect to } h \\
&= \cos x\cdot 0-\sin x\cdot 1 && \text{by the first and second Fundamental Trig Limits} \\
&= -\sin x
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} && \text{by the definition of } \tan x \\
&= \frac{(\frac{d}{dx}\sin x)(\cos x)-(\frac{d}{dx}\cos x)(\sin x)}{(\cos x)^2} && \text{by the Quotient Rule} \\
&= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} && \text{by the derivatives of sine and cosine} \\
&= \frac{1}{\cos^2 x} && \text{by the Trigonometric Pythagorean Theorem} \\
&= \sec^2 x && \text{by the definition of } \sec x
\end{aligned}$$

$$
\begin{aligned}
\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x} && \text{by the definition of } \tan x \\
&= \frac{(\frac{d}{dx}1)(\cos x) - (\frac{d}{dx}\cos x)(1)}{(\cos x)^2} && \text{by the Quotient Rule} \\
&= \frac{\sin x}{\cos^2 x} && \text{by the derivatives of 1 and cosine} \\
&= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \\
&= \sec x \tan x && \text{by the definition of } \sec x \text{ and } \tan x
\end{aligned}
$$

(There is also an easier proof using power rule and Chain Rule which I will accept but not require, as Chain Rule is in Unit 3.)

$$
\begin{aligned}
\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{\cos x}{\sin x} && \text{by the definition of } \cot x \\
&= \frac{(\frac{d}{dx}\cos x)(\sin x) - (\frac{d}{dx}\sin x)(\cos x)}{(\sin x)^2} && \text{by the Quotient Rule} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} && \text{by the derivatives of sine and cosine} \\
&= \frac{-1}{\sin^2 x} && \text{by the Trigonometric Pythagorean Theorem} \\
&= -\csc^2 x && \text{by the definition of } \csc x
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\csc x &= \frac{d}{dx}\frac{1}{\sin x} && \text{by the definition of } \tan x \\
&= \frac{(\frac{d}{dx}1)(\sin x) - (\frac{d}{dx}\sin x)(1)}{(\sin x)^2} && \text{by the Quotient Rule} \\
&= \frac{-\cos x}{\sin^2 x} && \text{by the derivatives of 1 and sine} \\
&= -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} \\
&= \csc x \cot x && \text{by the definition of } \csc x \text{ and } \cot x
\end{aligned}
$$

(There is also an easier proof using power rule and Chain Rule which I will accept but not require, as Chain Rule is in Unit 3.)