Volumes

1. (Calculator permitted) Let R be the region bounded by the curves $y = \sqrt{1+x^3}$ and $y = x+1$. Set up integrals for the following, but do not evaluate.

   (a) The area of R

   > **Solution:** The curves intersect at $(-1,0)$, $(0,1)$, and at $(2,3)$.
   >
   > $$\int_{-1}^{0} [\sqrt{1+x^3} - (x+1)]\, dx + \int_{0}^{2} [(x+1) - \sqrt{1+x^3}]\, dx$$

   (b) The volume of a solid whose base is R and cross-sections perpendicular to the x-axis are squares

   > **Solution:**
   >
   > $$\int_{-1}^{2} (\sqrt{1+x^3} - (x+1))^2\, dx$$

   (c) The volume of a solid whose base is R and cross-sections perpendicular to the y-axis are equilateral triangles

   > **Solution:**
   >
   > $$\int_{0}^{3} \frac{\sqrt{3}}{4} (\sqrt[3]{y^2-1} - (y-1))^2\, dx$$

   (d) The volume of the solid of revolution obtained by revolving R about the x-axis

   > **Solution:**
   >
   > $$\pi \int_{-1}^{0} [(\sqrt{1+x^3})^2 - (x+1)^2]\, dx + \pi \int_{0}^{2} [(x+1)^2 - (\sqrt{1+x^3})^2]\, dx$$

2. (Calculator permitted) Let R be the region bounded by the curves $y = e^x$ and $y = 2x + 1$.

(a) Find the area of R.

Solution: The curves intersect at $(0, 1)$ and at about $(1.25643, 3.51286)$.

$$\int_0^{1.25643} [(2x + 1) - e^x]\, dx \approx 0.322$$

(b) Find the volume of the solid of revolution obtained by revolving R about the x-axis.

Solution:

$$\int_0^{1.25643} \pi[(2x + 1)^2 - (e^x)^2]\, dx \approx 4.361$$

(c) Find the volume of the solid of revolution obtained by revolving R about the y-axis.

Solution:

$$\int_1^{3.51286} \pi[(\ln y)^2 - (\tfrac{1}{2}(y - 1))^2]\, dy \approx 1.324$$

(d) Find the volume of the solid of revolution obtained by revolving R about the line $y = 4$.

Solution:

$$\int_0^{1.25643} \pi[(4 - e^x)^2 - [4 - (2x + 1)]^2]\, dx \approx 3.737$$

3. (Calculator permitted, but as a challenge you may try without a calculator) Let R be the region bounded by the curves $y = x + 1$, $y = \frac{x}{2} + 1$, and $y = 4 - x$.

   (a) Find the area of R.

Solution: The curves intersect at $(0, 1)$, $(3/2, 5/2)$, and at $(2, 2)$.

$$\begin{aligned}
&\int_0^{3/2} \left[(x+1) - \left(\frac{x}{2}+1\right)\right] dx + \int_{3/2}^{2} \left[(4-x) - \left(\frac{x}{2}+1\right)\right] dx \\
&= \int_0^{3/2} \frac{x}{2}\, dx + \int_{3/2}^{2} \left(3 - \frac{3}{2}x\right) dx \\
&= \frac{1}{4}x^2\Big|_0^{3/2} + \left(3x - \frac{3}{4}x^2\right)\Big|_{3/2}^{2} \\
&= \frac{9}{16} + 3\cdot\frac{1}{2} - \frac{3}{4}\cdot\left(4 - \frac{9}{4}\right) \\
&= \frac{9}{16} + \frac{24}{16} - \frac{21}{16} = \frac{12}{16} = \frac{3}{4}
\end{aligned}$$

   (b) Find the volume of the solid of revolution obtained by revolving R about the x-axis.

Solution:

$$\begin{aligned}
&\int_0^{3/2} \pi\left[(x+1)^2 - \left(\frac{x}{2}+1\right)^2\right] dx + \int_{3/2}^{2} \pi\left[(4-x)^2 - \left(\frac{x}{2}+1\right)^2\right] dx \\
&= \pi\int_0^{3/2} \left(\frac{3}{4}x^2 + x\right) dx + \pi\int_{3/2}^{2} \left(\frac{3}{4}x^2 - 9x + 15\right) dx \\
&= \pi\left(\frac{1}{4}x^3 + \frac{1}{2}x^2\right)\Big|_0^{3/2} + \pi\left(\frac{1}{4}x^3 - \frac{9}{2}x^2 + 15x\right)\Big|_{3/2}^{2} \\
&= \pi\left[\frac{27}{32} + \frac{9}{8} + \frac{1}{4}\cdot\left(8 - \frac{27}{8}\right) - \frac{9}{2}\cdot\left(4 - \frac{9}{4}\right) + 15\cdot\left(2 - \frac{3}{2}\right)\right] \\
&= \pi\left[\frac{63}{32} + \frac{37}{32} - \frac{63}{8} + \frac{15}{2}\right] = \frac{63}{32}\pi + \frac{25}{32}\pi = \frac{11}{4}\pi
\end{aligned}$$

(c) Find the volume of the solid of revolution obtained by revolving R about the y-axis.

Solution:

$$\int_1^2 \pi\left[(2y-2)^2-(y-1)^2\right]\,dy+\int_2^{5/2}\pi\left[(4-y)^2-(y-1)^2\right]\,dy$$
$$=\pi\int_1^2\left(3y^2-6y+3\right)\,dx+\pi\int_{3/2}^2(-6y+15)\,dx$$
$$=\pi\left(y^3-3y^2+3y\right)\Big|_1^2+\pi\left(15y-3y^2\right)\Big|_2^{5/2}$$
$$=\pi\left[(8-1)-3(4-1)+3(2-1)+15\cdot\frac{1}{2}-3\cdot\left(\frac{25}{4}-4\right)\right]$$
$$=\pi\left[1+\frac{30}{4}-\frac{27}{4}\right]=\frac{7}{4}\pi$$

(d) (*) There exists a real number k such that if we revolve R about the line $x = k$, the resulting solid has the same volume as the solid obtained by revolving R about the x-axis. Find k.

Solution:

$$\int_1^2 \pi\left[(2y-2-k)^2-(y-1-k)^2\right]\,dy+\int_2^{5/2}\pi\left[(4-y-k)^2-(y-1-k)^2\right]\,dy$$
$$=\pi\int_1^2\left(3y^2-(6+2k)y+3+2k\right)\,dx+\pi\int_{3/2}^2((4k-6)y+15-10k)\,dx$$
$$=\pi\left(y^3-(3+k)y^2+(3+2k)y\right)\Big|_1^2+\pi\left((15-10k)y+(2k-3)y^2\right)\Big|_2^{5/2}$$
$$=\pi\left[(8-1)-(3+k)(4-1)+(3+2k)(2-1)+(15-10k)\cdot\frac{1}{2}+(2k-3)\cdot\left(\frac{25}{4}-4\right)\right]$$
$$=\pi\left[1-k+\frac{30}{4}-5k-\frac{27}{4}+\frac{9}{2}k\right]=\frac{7-6k}{4}\pi$$

Using our answer for part b), we need

$$\frac{7-6k}{4}\pi=\frac{11}{4}\pi$$
$$k=-\frac{2}{3}$$