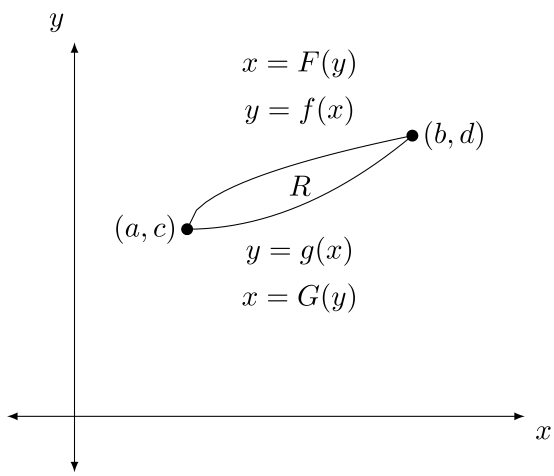
Volumes

Suppose R is the region bounded by the curves as indicated in the below. Assume $a < b$ and $c < d$.



Set up integrals for the following, but do not evaluate.

1. The volume of a solid with base given by R and whose cross-sections perpendicular to:

   (a) the x-axis are squares.

   Solution:

   $$\int_a^b s^2\,dx = \int_a^b [f(x) - g(x)]^2\,dx$$

   (b) the x-axis are rectangles whose heights are half their width.

   Solution:

   $$\int_a^b \frac{1}{2}s^2\,dx = \int_a^b \frac{1}{2}[f(x) - g(x)]^2\,dx$$

   (c) the x-axis are right isosceles triangles whose hypotenuse is along the base.

   Solution:

   $$\int_a^b \frac{1}{4}s^2\,dx = \int_a^b \frac{1}{4}[f(x) - g(x)]^2\,dx$$

(d) the x-axis are semi-circles.

Solution:

$$\int_a^b \frac{1}{2}\pi \left(\frac{s}{2}\right)^2 \, dx = \int_a^b \frac{\pi}{8}[f(x) - g(x)]^2 \, dx$$

(e) the y-axis are semi-circles.

Solution:

$$\int_c^d \frac{1}{2}\pi \left(\frac{s}{2}\right)^2 \, dy = \int_c^d \frac{\pi}{8}[G(y) - F(y)]^2 \, dy$$

(f) the y-axis are rectangles whose heights are three times their width.

Solution:

$$\int_c^d 3s^2 \, dy = \int_c^d 3[G(y) - F(y)]^2 \, dy$$

(g) the y-axis are right isosceles triangles whose hypotenuse is not on the base.

Solution:

$$\int_c^d \frac{1}{2}s^2 \, dy = \int_c^d \frac{1}{2}[G(y) - F(y)]^2 \, dy$$

(h) the y-axis are triangles whose heights are three times their base.

Solution:

$$\int_c^d \frac{3}{2}s^2 \, dy = \int_a^b \frac{3}{2}[G(y) - F(y)]^2 \, dy$$

2. The volume of the solid of revolution where we revolve the region R about:

(a) the x-axis

Solution:

$$\pi \int_a^b (r_{\text{outer}}^2 - r_{\text{inner}}^2)\, dx = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)\, dx$$

(b) the y-axis

Solution:

$$\pi \int_c^d (r_{\text{outer}}^2 - r_{\text{inner}}^2)\, dy = \pi \int_c^d ([G(y)]^2 - [F(y)]^2)\, dy$$

(c) the line $y = -2$ (assume $c > -2$)

Solution:

$$\begin{aligned}\pi \int_a^b (r_{\text{outer}}^2 - r_{\text{inner}}^2)\, dx &= \pi \int_a^b ([f(x) - (-2)]^2 - [g(x) - (-2)]^2)\, dx \\ &= \pi \int_a^b ([f(x) + 2]^2 - [g(x) + 2]^2)\, dx\end{aligned}$$

(d) the line $x = -3$ (assume $a > -3$)

Solution:

$$\begin{aligned}\pi \int_c^d (r_{\text{outer}}^2 - r_{\text{inner}}^2)\, dy &= \pi \int_c^d ([G(y) - (-3)]^2 - [F(y) - (-3)]^2)\, dy \\ &= \pi \int_c^d ([G(y) + 3]^2 - [F(y) + 3]^2)\, dy\end{aligned}$$

(e) the line $y = 4$ (assume $d < 4$)

Solution:

$$\pi \int_a^b (r_{\text{outer}}^2 - r_{\text{inner}}^2)\, dx = \pi \int_a^b ([4 - g(x)]^2 - [4 - f(x)]^2)\, dx$$

(f) the line $x = 5$ (assume $b < 5$)

Solution:

$$\pi \int_c^d (r_{\text{outer}}^2 - r_{\text{inner}}^2)\, dy = \pi \int_c^d ([5 - F(y)]^2 - [5 - G(y)]^2)\, dy$$