π is irrational
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Note: The following proof is due to Niven (1947).
Here is an outline of what we will do:

1. Assume π is rational, so $\pi = a/b$ for some positive integers a and b.
2. Define a function $f(x)$ which only depends on constants a and b.
3. Prove that $\displaystyle\int_0^\pi f(x)\sin x\,dx$ is a positive integer.
4. Prove that $\displaystyle\int_0^\pi f(x)\sin x\,dx$ is less than 1.
5. Conclude by contradiction that π is irrational.

We use the definition: π is the smallest positive root of $\sin x$. We prove π is irrational. Suppose otherwise, that $\pi = a/b$ for positive integers a, b. For $n \in \mathbb{N}$, define the polynomial

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

where n will be defined later. Define also the polynomial

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

Lemma 1. $F(0) \in \mathbb{Z}$

Proof. If $k < n$, then the coefficient of x^k in $f(x)$ is 0, since the x^n factor guarantees a degree of at least n. If $n \le k \le 2n$, then the coefficient of x^k is $c_k/n!$ for some integer c_k, coming from the binomial expansion since $a, b \in \mathbb{Z}$. Then

$$f^{(k)}(0) = \begin{cases} 0 & \text{if } 0 \le k < n \\ \frac{k!c_k}{n!} & \text{if } n \le k \le 2n \end{cases}$$

In particular, since $k!/n! \in \mathbb{Z}$ for $n \le k \le 2n$, we have $f^{(k)}(0) \in \mathbb{Z}$ for $1 \le k \le 2n$. Therefore, $F(0) \in \mathbb{Z}$. $\square$

Lemma 2. $F(\pi) \in \mathbb{Z}$

Proof. Notice

$$\begin{aligned} f(\pi - x) &= \frac{(\pi - x)(a - b(\pi - x))^n}{n!} = \frac{(a/b - x)^n(a - b(a/b - x))^n}{n!} \\ &= \frac{(a/b - x)^n(bx)^n}{n!} = \frac{(a - bx)^n x^n}{n!} = f(x) \end{aligned}$$

By the chain rule (applied k times),

$$\begin{aligned} &f^{(k)}(\pi - x) = (-1)^k f^{(k)}(x) \\ \implies &f^{(k)}(\pi) = (-1)^k f^{(k)}(0) \in \mathbb{Z} \\ \implies &F(\pi) \in \mathbb{Z} \end{aligned}$$

$\square$

Lemma 3.

$$\int_0^\pi f(x) \sin x \, dx = F(\pi) - F(0)$$

Proof. Since $f(x)$ is a polynomial of degree $2n$, we have that $f^{(2n+2)}(x) = 0$, and so

$$\begin{aligned} F''(x) + F(x) &= [f^{(2)}(x) - f^{(4)}(x) + \cdots + (-1)^{n+1} f^{(2n)}(x)] \\ &\quad + [f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)] \\ &= f(x) \end{aligned}$$

By the product rule,

$$\begin{aligned} \frac{d}{dx}(F'(x)\sin x - F(x)\cos x) &= F''(x)\sin x + F'(x)\cos x - F'(x)\cos x + F(x)\sin x \\ &= F''(x)\sin x + F(x)\sin x \\ &= f(x)\sin x \end{aligned}$$

Then by the Fundamental Theorem of Calculus,

$$\int_0^\pi f(x)\sin x \, dx = (F'(x)\sin x - F(x)\cos x)\Big|_0^\pi = F(\pi) - F(0)$$

where we have used the fact that $\sin(0) = \sin(\pi) = 0$, $\cos(0) = 1$, and $\cos(\pi) = -1$. (Notice $f(x)\sin x$ is continuous, so this is a proper integral.) $\square$

By Lemmas 1 through 3, $\int_0^\pi f(x)\sin x \, dx$ is an integer. Furthermore, $f(x) > 0$ and $\sin x > 0$ for $0 < x < \pi$ (if concerned about the assumption $a > b$, then restrict to even n). Thus $\int_0^\pi f(x)\sin x \, dx$ is a positive integer.

On the other hand, for $0 < x < \pi$, we have $0 < x(a - bx) < \pi a$ and $0 < \sin x < 1$, so

$$\int_0^\pi f(x)\sin x \, dx \leq \int_0^\pi \frac{(\pi a)^n}{n!} \, dx = \pi \frac{(\pi a)^n}{n!}$$

For sufficiently large n, this is less than 1, contradicting the fact that this integral is a positive integer. Therefore, π is irrational.